A cosmological model with *time varying* cosmological constant

Yousef Sobouti and M. B. Jahani Poshteh
Institute for Advanced Studies in Basic Sciences (IASBS) No. 444,
Prof. Yousef Sobouti Blvd. P. O. Box 45195-1159 Zanjan Iran Postal Code 45137-66731
email: sobouti@iasbs.ac.ir, mb.jahani@iasbs.ac.ir

A time dependent cosmological constant violates the conservation of energy and momenta. Models as such exhibit certain peculiarities. Only a combination of (density + pressure) remains in their final formulation, leaving no room for an equation of state to play a role. Nevertheless, one may build model universes that expand at accelerating rates.
Key words: variable cosmological constant, Friedmann universes, accelerating universe

PACS numbers:

I. INTRODUCTION

One of the tenets of the standard model of cosmology is the conservation of the energy momentum tensor. In fact, the field equation of Einstein,

$$G^{\mu\nu} - \lambda g^{\mu\nu} = -8\pi G T^{\mu\nu}, \quad \lambda \text{ const.}, \tag{1.1}$$

is so designed to yield

$$G^{\mu\nu}{}_{;\nu} = 0, \quad T^{\mu\nu}{}_{;\nu} = 0. \tag{1.2}$$

It is true that there are ample evidence to support conservation of the energy and momentum, Eq.(1.2), at the present epoch and in vast cosmological expanses; but there are no cogent reasons and/or convincing observations to extrapolate the concept to all cosmological times and to the whole Universe.

On the other hand, Einstein's cosmological constant, originally introduced to be an all time constant, in recent years and for certain purposes, is speculated to be a time dependent parameter. Linde [1] considers a variable λ in connection with spontaneous symmetry breakings in QFT. Kastor [2] invokes it in his study of the thermodynamics of black holes. One of us (JP) [3] also has felt the need for a variable cosmological constant in his study of the thermodynamics of Horova-Lifshitz' quantum gravity. A time dependent $\lambda(t)$, however, violates the second of Er.(1.2). Certain peculiarities develop not encountered in the standard models and, to the best of our knowledge, not addressed in the literature.

Here we analyse a perfect fluid-filled Friedmann universe in the presence of a time dependent cosmological constant (never mind calling a variable a constant). The work is not claimed to be a finished one. It is presented with the hope of inviting comments from the community of experts.

II. EXPOSITION OF THE PROBLEM

Let $\lambda(t)$ be a function of time and take the 4-divergence of Eq.(1.1). Considering that $G^{\mu\nu}{}_{;\nu} = 0$, one obtains

$$\lambda_{,\nu} g^{\mu\nu} = 8\pi G T^{\mu\nu}{}_{;\nu}. \tag{2.1}$$

In a Friedmann universe with a perfect fluid content, one has

$$ds^2 = -dt^2 + R(t)^2 \left[\frac{dr^2}{1-kr^2} + r^2 d\Omega^2 \right], \tag{2.2}$$

$$T^{\mu\nu} = p g^{\mu\nu} + (\rho + p) U^\mu U^\nu, \quad U^\mu : (1,0,0,0). \tag{2.3}$$

Equation(2.1) reduces to

$$\dot{\lambda} = -8\pi G \left[\dot{\rho} + 3\frac{\dot{R}}{R}(\rho + p) \right].$$

Upon integration one obtains

$$\lambda = \pm \frac{3}{\theta^2} - 8\pi G \left[\rho + 3 \int^t \frac{\dot{R}}{R}(\rho + p) dt \right], \tag{2.4}$$

where $\pm 3\theta^{-2}$ is an integration constant (a constant cosmological constant, so written for later convenience). Defined as such, θ has the dimension of time. Substituting Eq.(2.4) in Eq.(1.1) and writing out its *time-time* and *space-space* components gives

$$\frac{1}{R^2}(\dot{R}^2 + k) = \pm \frac{1}{\theta^2} - 8\pi G \int^t \frac{\dot{R}}{R}(\rho + p) dt, \tag{2.5}$$

$$\frac{1}{R^2}(2R\ddot{R} + \dot{R}^2 + k) = \pm \frac{3}{\theta^2} - 8\pi G \left[(\rho + p) + 3 \int^t \frac{\dot{R}}{R}(\rho + p) dt \right] \tag{2.6}$$

Taking the time derivative of Eq.(2.5) gives

$$\frac{1}{R^2}(R\ddot{R} - \dot{R}^2 - k) = -4\pi G(\rho + p). \tag{2.7}$$

The three Eqs.(2.5) - (2.7) are, however, not independent. Adding three times of Eq.(2.5) to two times of Eq.(2.7) gives Eq.(2.6). Only the combination $(\rho + p)$ appears in these equations. This is in contrast to the standard perfect fluid-filled cosmological models, in which ρ and p play independent roles, and allow one to invite in an equation of state, $p(\rho)$, and have an additional degree of freedom to maneuver. Evidently, this freedom is used up in the process of expressing λ in terms of ρ and p, Eq.(2.4).

With the assumptions made so far, there is no clue as to how ρ, p, $(\rho + p)$, or for that matter the Hubble parameter and deceleration parameters, and the expansion factor evolve in time. A new physical input is needed to proceed further. One convenient provision is to assume

$$4\pi G(\rho + p) = \frac{1}{\tau^2} f(R), \quad \frac{1}{\tau^2} = 4\pi G(\widehat{\rho + p}), \tag{2.8}$$

where $f(R)$ is a dimensionless function of R to be decided as one may desire and τ is a characteristic time of the model, to be calculated from some mean density and pressure of the model. For $(\rho+p) = 10^{-29}\text{gr/cm}^3$, about the density of the actual Universe, one finds

$$\tau = 11 \times 10^9 \text{ yr.}$$

Substituting Eq.(2.8) in Eq.(2.5), one arrives at a non-linear integro-differential equation for R . Thus,

$$\left(\frac{\dot R}{R}\right)^2 + \frac{k}{R^2} = \pm\frac{1}{\theta^2} - \frac{2}{\tau^2}\int^t \frac{\dot R}{R} f(R) dt. \tag{2.9}$$

For whatever it may be worth, below we examine three examples for $k = 0$ and various assumptions for θ^{-2} and $f(R)$, just to demonstrate how one may proceed to construct one's own toy models.

Example 1. In Eq.(2.9 let $k = 0$, choose $+\theta^{-2}$, and assune $f(R) = 1$, i.e. a constant $(\rho + p)$ (one may wish to call the model a steady state universe, of course, not in the sense that Hoyle and Narlikar have used the term before [4]). Further let

$$F = \int^t \frac{\dot R}{R} dt, \quad \frac{\dot R}{R} = \dot F.$$

Equation(2.9) reduces to

$$\dot F^2 = \theta^{-2} - 2F/\tau^2, \quad \dot F = \pm(\theta^{-2} - 2F/\tau^2)^{1/2}. \tag{2.10}$$

Upon integration of Eq.(2.10) one obtains F, R, the Hubble parameter, $H = \dot R/R$, and the deceleration parameter, $q = -R\ddot R/\dot r^2$:

$$F = \frac{1}{2}[\tau^2/\theta^2 - (t_0 - t)^2/\tau^2], \tag{2.11}$$
$$R = R_0 \exp[-(t_0 - t)^2/2\tau^2], \tag{2.12}$$
$$H = (t_0 - t)/\tau^2 = \dot F = \dot R/R, \tag{2.13}$$
$$q = [\tau^2/(t_0 - t)^2 - 1], \tag{2.14}$$

where $t = 0$ represents the present epoch. The integration constant, t_0, may be determined from Eq.(2.13) in terms of the Hubble constant at the present epoch. Thus, for $H_0 = 100h$ km/s Mps one finds

$$t_0 = H_0\tau^2 \approx 12h \times 10^9 \text{ yrs.}$$

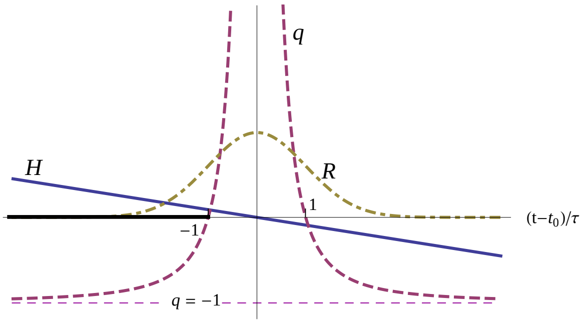

FIG. 1: Hubble parameter H, Deceleration parameter, q, and Scale Factor, R, as functions of time for a constant $(\rho+p)$. In the range $t < t_0 - \tau$, the model has an accelerating expansion. Scale factor is Gaussian. There is no big bang. The model begins from zero size in the infinitely remote past and fades away in the infinitely remote future.

The Hubble parameter is linear in time, positive in $t < t_0$ and negative otherwise. The scale factor, R, is Gaussian in time with a characteristic e-folding time $\sqrt{2}\tau$. It has been zero in the infinitely remote past and will be so in the infinitely remote future. The model will expand to its maximum size R_0 in the future time $t = t_0$. The deceleration parameter, q, is negative (i.e. expansion accelerates) if $|t - t_0| > \tau$. Figure 1 is a plot of R, H and q versus time. The time span during which H is positive and q negative (i.e. the model has an accelerating expansion) is marked by the thick black line. The plot of q has two asymptotes: 1) $q \to -1$ as $(t - t_0) \to \pm\infty$, and $q \to \infty$ as $(t - t_0) \to 0$.

Example 2. In Eq.(2.9) let $k = 0$, choose $+\theta^{-2}$, assume $f(R) = R^{-1}$, i.e. a decreasing $(\rho + p)$proportional R^{-1}, and obtain:

$$\frac{\dot R^2}{R^2} = \frac{1}{\theta^2} + \frac{2}{\tau^2}\frac{1}{R} = \frac{1}{\theta^2}\left[1 + 2\frac{R_0}{R}\right], \tag{2.15}$$

where $R_0 = (\theta/\tau)^2$, and a possible integration constant in integrating $\int dRf(R)/R$ is absorbed in the redefined θ. Solutions to Eq.(2.15) are

$$R = R_0[\cosh(t - t_0)/\theta - 1], \tag{2.16}$$
$$H = \frac{1}{\theta}\frac{\sinh(t - t_0)/\theta}{\cosh(t - t_0)/\theta - 1}, \tag{2.17}$$
$$q = -\frac{\cosh(t - t_0)/\theta}{\cosh(t - t_0)/\theta + 1}. \tag{2.18}$$

Again t_0 is an integration constant. At $t = 0$ one has

$$H_0 = -\frac{1}{\theta}\frac{\sinh(t_0/\theta)}{\cosh(t_0/\theta) - 1}, \tag{2.19}$$
$$q_0 = -\frac{\cosh(t_0/\theta)}{\cosh(t_0/\theta) + 1}. \tag{2.20}$$

Given H_0 and q_0 one may find θ and t_0/θ. Thus

$$\frac{t_0}{\theta} = \cosh^{-1}(\frac{q_0}{q_0 + 1}),$$

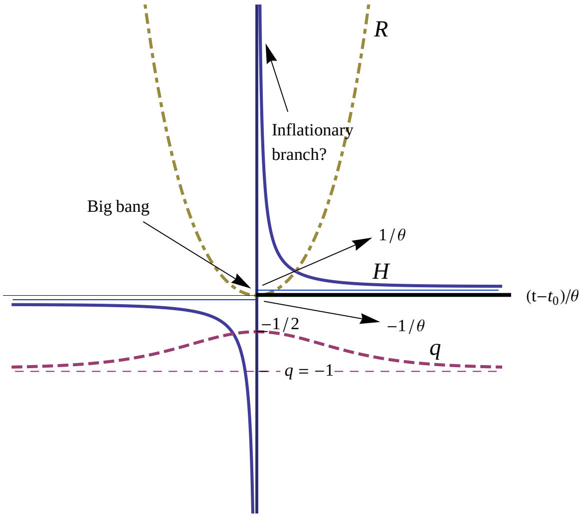

FIG. 2: Hubble parameter, H, deceleration parameter, q, and Expansion factor, R, versus time for $(\rho+p) \propto R^{-1}$ and $+\theta^2$ term. Deceleration is an all time negative. Hubble parameter is positive in the range $(t-t_0) > 0$. Big bang begins at $(t-t_0) = 0$. Vicinity of the infinite asymptote of H is suggested as an inflationary phase. Plots are symmetric about $(t-t_0) = 0$ axis, suggesting a time reversed evolutionary past much the same as the evolution to the futre.

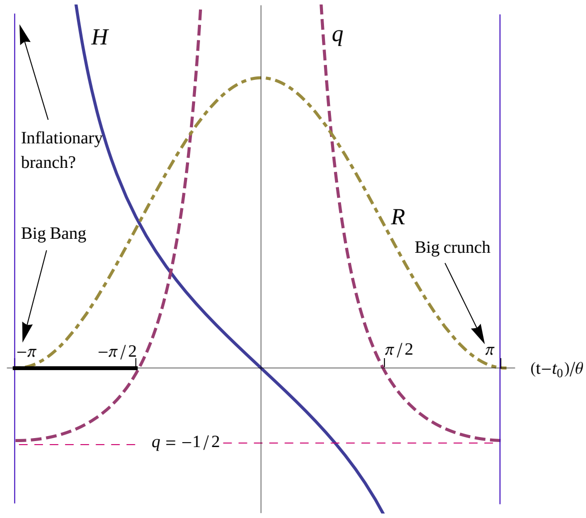

FIG. 3: Hubble parameter, H, deceleration parameter, q, and scale Factor, R, as functions of time for $(\rho+p) \propto R^{-1}$ and $-\theta^{-2}$ term. The model is periodic in cosmic time. The model has an accelerating expansion in the range $-\pi/2 < (t-t_0)/\theta < 0,$. Big bang and big crunch occur in $(t-t_0)/\theta = -\pi/2$ and $3\pi/2$, respectively.

$$\theta^2 = \frac{1}{H_0^2}(2|q_0|-1) > 0, \quad \text{realized for} \quad |q| > 0.5.$$

Figure 2 is a plot of H, q, and R, as functions of time, for $(\rho+p) \propto R^{-1}$. The deceleration parameter is an all time negative and falls in range $-1 \le q \le -1/2$. The time span during which H is positive and q negative is $(t-t_0) \ge 0$, and is marked by the thick black line. Big bang, $R = 0$, begins at $(t-t_0) = 0$ with an infinite expansion rate (can one call this an inflationary phase? The e-fold droping time of H, however, is θ, not perhaps short enough to justify the common usage of the term inflation). The plots are symmetric about $(t-t_0) = 0$ axis, suggesting a time reversed evolutionary past much the same as the evolution to the future.

Example 3. In Eq.(2.9) let $k = 0$, chooe $-\theta^{-2}$ term, assume $f(R) = R^{-1}$, a decreasing $(\rho+p)$, and obtain:

$$\frac{\dot{R}^2}{R^2} = \frac{2}{\tau^2}\frac{1}{R} - \frac{1}{\theta^2} = \frac{1}{\theta^2}\left[2\frac{R_0}{R} - 1\right], \tag{2.21}$$

where, again, $R_0 = (\theta/\tau)^2$, and a possible integration constant in integrating $\int dRf(R)/R$ is absorbed in the redefined θ. Solutions to Eq.(2.21) are

$$R = R_0[1+\cos(t-t_0)/\theta], \tag{2.22}$$

$$H = -\frac{1}{\theta}\frac{\sin(t-t_0)/\theta}{1+\cos(t-t_0)/\theta}, \tag{2.23}$$

$$q = \frac{\cos(t-t_0)/\theta}{1-\cos(t-t_0)/\theta}. \tag{2.24}$$

At $t = 0$ one has

$$H_0 = \frac{1}{\theta}\frac{\sin(t_0/\theta)}{1+\cos(t_0/\tau)}, \tag{2.25}$$

$$q_0 = \frac{\cos(t_0/\tau)}{1-\cos(t_0/\tau)}. \tag{2.26}$$

Equation(2.25) enables one to estimate t_0/τ from the present day Hubble constant of the model. Thus,

$$\frac{t_0}{\theta} = \cos^{-1}\left(\frac{q_0}{1+q_0}\right),$$

$$\theta^2 = 1 + 2q_0, \quad \text{realized for} \quad q_0 < 0.5.$$

Figure 3 is a plot of H, q, and R, as functions of time, for $(\rho+p) \propto R^{-1}$. The model is periodic in cosmic time. The minimum value of the Deceleration parameter is -0.5. It has $\pm\infty$ asymptotes at $(t-t_0)/\theta \to 0,\ \pi,\ 2\pi,\ \ldots$. In the range $-\pi/2 < \Omega(t-t_0) < 0$, there is an accelerating expansion. Big bang and big crunch occur in $\Omega(t-t_0) = -\pi/2$ and $3\pi/2$, respectively.

III. CONCLUDING REMARKS

A time varying cosmological constant violates the energy momentum conservation. Only the combination $(\rho+p)$ appears in the final conclusions. The independent identity of ρ and p and along with it the need for an equation of state is lost in the process. A new assumption as to how $(\rho+p)$ evolves in time is required.

One has the freedom to design one's own toy models, by choosing appropriate input data and integration constants, to mimic the actual Universe to one's desire. Three examples are examined here:

i. $k = 0$, $+\theta^{-2}$ term, and $(\rho + p)$ constant,

ii. $k = 0$, $+\theta^{-2}$ term, and $(\rho + p) \propto R^{-1}$,

iii. $k = 0$, $-\theta^{-2}$ term, and $(\rho + p) \propto R^{-1}$.

All three have analytical solutions and all have time intervals in which the models expand at accelerating rates.

---

[1] A. D. Linde, *JETP Lett.* **19**, 183, 1974.
[2] D. Kastor, S.Ray, and J. Traschen, *Class. Quant. Grav.*, **26**, 195011, 2009.
[3] M. B. Jahani Poshteh and N. Riazi, in preparation, 2017.
[4] F. Hoyle, J. V. Narlikar, *Proc. Roy. Soc.*, **A277**, 1; **A282**, 178, 184,191, 1964.