SOME ANALYTICAL RESULTS IN DYNAMICS OF SPHEROIDAL GALAXIES

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Abstract. The surfaces of section in a harmonic oscillator potential, perturbed by quartic terms, are obtained analytically. A succession of action-angle, Lissajous and Lie transformations near the 1:1 commensurability, reduces the three-dimensional motion to a one-dimensional one. The latter is solved in terms of Jacobi's elliptic functions. Existence conditions for periodic orbits are found and two general families of such solutions are introduced. Two examples of regular motions in oblate and prolate spheroids are discussed.

Key words: galactic dynamics, Lissajous transformation, elliptic functions

1. Introduction

Internal dynamics of galactic systems has been an attractive field of research in recent years. Perturbed harmonic oscillators in two dimensions, have been investigated by Deprit and Elipe [1], Miller [2], Caranicolas [3], Elipe et al. [4] and Contopoulos and Polymilis [5]. Numerically constructed Poincaré maps and mapping models like that of Wisdom [6] have been used in these papers. In certain circumstances, it has also been possible to integrate the equations of motion analytically. For example, by a transformation to Hopf's coordinates, Deprit and Elipe [1] were able to solve their normalized Hamiltonian system by elliptic functions.

In three dimensions, regular periodic orbits inside elliptical galaxies have been considered by Davoust [7, 8] and Jalali et al. [9]. They have used the Poincaré–Lindstedt and the implicit function methods, respectively. The existence conditions of periodic orbits at exact resonances along with the initial conditions leading to such solutions have been determined. This paper further considers three-dimensional harmonic oscillators perturbed with quartic potentials and provides analytical solutions for motions near the 1:1 resonance. In Sections 2, 3 and 4 we integrate the equations of motion by quadratures. In Section 5 we investigate possible types of periodic and quasi-periodic orbits and their surrounding tubes. In Section 6 we study local stability of oblate and prolate galaxies. We show that equatorial motions in oblate galaxies

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are locally stable as long as the perturbative analytical scheme remains valid, while in prolate systems they can be unstable.

2. Quartic Hamiltonians and Normalization by Lie Transformations

We consider a spheroidal galaxy with a quadratic mass distribution, $\rho = \rho_0[1 - (x^2 + y^2)/a^2 - z^2/c^2]$. The corresponding potential, as given by Lebovitz [10], is quartic in (x, y, z). The Hamiltonian of a mass point moving in such a potential is given by Davoust [8],

$$H = H_0 + \epsilon H_1, \tag{1a}$$

$$H_0 = \tfrac{1}{2}(X^2 + Y^2 + Z^2) + \tfrac{1}{2}\omega^2(x^2 + y^2) + \tfrac{1}{2}\lambda^2 z^2 + \Omega(xY - yX), \tag{1b}$$

$$H_1 = -\tfrac{1}{4}\gamma z^4 - \tfrac{1}{4}\alpha(x^2 + y^2)^2 - \tfrac{1}{2}\beta(x^2 + y^2)z^2, \tag{1c}$$

$$X = \dot{x} + \Omega y, \quad Y = \dot{y} - \Omega x, \quad Z = \dot{z}, \tag{1d}$$

where ϵ is a perturbation parameter and will eventually be put equal to one, a and c are the semi-axes of the spheroid, Ω is the solid-body rotation frequency of the galaxy, λ is the unperturbed orbital frequency along the z-axis and ω is the unperturbed frequency associated with the elliptic anomaly of the x–y oscillations. It should be noted that the z and x–y oscillations are coupled via the parameter β. From [10], the constant parameters, α, β, γ, ω and λ are functions of the eccentricity of the galaxy through the following relations

$$\begin{aligned}
\omega^2 &= \int_0^\infty \frac{e^3\,\mathrm{d}u}{(\delta_1 + u)^2\sqrt{\delta_1 + u}}, \qquad \lambda^2 = \int_0^\infty \frac{e^3\,\mathrm{d}u}{(\delta_1 + u)(\delta_2 + u)\sqrt{\delta_2 + u}}, \\
\alpha &= \int_0^\infty \frac{e^5\,\mathrm{d}u}{(\delta_1 + u)^3\sqrt{\delta_2 + u}}, \qquad \beta = \int_0^\infty \frac{e^5\,\mathrm{d}u}{(\delta_1 + u)(\delta_2 + u)\sqrt{\delta_2 + u}}, \\
\gamma &= \int_0^\infty \frac{e^5\,\mathrm{d}u}{(\delta_2 + u)^2(\delta_1 + u)\sqrt{\delta_2 + u}},
\end{aligned} \tag{2}$$

where

$$e^2 = 1 - \left(\frac{c}{a}\right)^2, \quad \delta_1 = 1, \quad \delta_2 = 1 - e^2; \quad \text{if} \quad c < a \ \text{(oblate spheroids)},$$

$$e^2 = 1 - \left(\frac{a}{c}\right)^2, \quad \delta_1 = 1 - e^2, \quad \delta_2 = 1; \quad \text{if} \quad c > a \ \text{(prolate spheroids)}.$$

The zero order Hamiltonian is fully integrable and has the following integrals

$$\omega L = \tfrac{1}{2}(X^2 + Y^2) + \tfrac{1}{2}\omega^2(x^2 + y^2), \tag{3a}$$

$$\Omega G = \Omega(xY - yX), \tag{3b}$$

$$\lambda K = \tfrac{1}{2}Z^2 + \tfrac{1}{2}\lambda^2 z^2. \tag{3c}$$

A generalized action-angle canonical transformation, $(l, g, k, L, G, K) \rightarrow (x, y, z, X, Y, Z)$, can be devised to express H_0 in its integrals, where (l, g, k) are the angle variables and (L, G, K) are their conjugate actions, respectively. Thus,

$$x = s\cos(g + l) - d\cos(g - l), \tag{4a}$$

$$y = s\sin(g + l) - d\sin(g - l), \tag{4b}$$

$$X = -\omega[s\sin(g + l) + d\sin(g - l)], \tag{4c}$$

$$Y = \omega[s\cos(g + l) + d\cos(g - l)], \tag{4d}$$

$$s^2 = \frac{1}{2\omega}(L + G), \qquad d^2 = \frac{1}{2\omega}(L - G), \tag{4e}$$

$$z = \sqrt{\frac{2K}{\lambda}}\sin k, \tag{5a}$$

$$Z = \sqrt{2K\lambda}\cos k. \tag{5b}$$

Equations (4) are the Lissajous transformations of Deprit [11]. The variables (l, g, k, L, G, K) are defined in the domain

$$D = [0, 2\pi[\times[0, 2\pi[\times[0, 2\pi[\times\{L > 0\} \times \{|G| \leqslant L\} \times \{K \geqslant 0\}. \tag{6}$$

The Hamiltonian transforms to

$$H_0 = \omega L + \Omega G + \lambda K, \tag{7a}$$

$$\begin{aligned} H_1 \;=\; & -\frac{1}{8}\left[\frac{\alpha}{\omega^2}(3L^2 - G^2) + \frac{4\beta}{\omega\lambda}KL + \frac{3\gamma}{\lambda^2}K^2\right] + \\ & + \frac{1}{2\lambda}\left(\frac{\beta}{\omega}L + \frac{\gamma}{\lambda}K\right)K\cos 2k - \frac{\gamma}{8\lambda^2}K^2\cos 4k + \\ & + \frac{1}{2\omega}\left(\frac{\alpha}{\omega}L + \frac{\beta}{\lambda}K\right)\sqrt{L^2 - G^2}\cos 2l - \frac{\alpha}{8\omega^2}(L^2 - G^2)\cos 4l - \\ & - \frac{\beta}{4\omega\lambda}K\sqrt{L^2 - G^2}\,[\cos(2k + 2l) + \cos(2k - 2l)]. \end{aligned} \tag{7b}$$

The angle g has become cyclic and G is an integral of the total Hamiltonian in addition to that of the zero order one. This reduces the six-dimensional phase space to a four-dimensional one.

The k angle describes the oscillations in z-direction, and l, the elliptic anomaly of the orbits, is that in the x–y plane. In nearly spherical galaxies, let $\mu = (\lambda - \omega)/\epsilon$ be small. Therefore, near the 1:1 commensurability the angle $(k - l)$ will vary slowly. Correspondingly, to separate the Hamiltonian into slowly and rapidly varying terms, we go to a rotating system about the z-axis with the angular frequency ω. This can be achieved by a canonical transformation $(p, q, P, Q) \to (l, k, L, K)$ through the generating function $S = (k - l)Q + lP$. One obtains

$$\begin{aligned} q &= \frac{\partial S}{\partial Q} = k - l, \qquad p = \frac{\partial S}{\partial P} = l, \\ K &= \frac{\partial S}{\partial k} = Q, \qquad L = \frac{\partial S}{\partial l} = P - Q. \end{aligned} \tag{8}$$

The transformed Hamiltonian, ignoring the constant terms in G, becomes

$$\begin{aligned} H_0 &= \omega P, \\ H_1 &= \mu Q - \frac{3\alpha}{8\omega^2}(P-Q)^2 - \frac{\beta}{2\omega\lambda}Q(P-Q) - \frac{3\gamma}{8\lambda^2}Q^2 + \\ &\quad + \frac{1}{2\lambda}\left[\frac{\beta}{\omega}(P-Q) + \frac{\gamma}{\lambda}Q\right] Q \cos 2(q+p) - \frac{\gamma}{8\lambda^2}Q^2 \cos 4(q+p) + \\ &\quad + \frac{1}{2\omega}\left[\frac{\alpha}{\omega}(P-Q) + \frac{\beta}{\lambda}Q\right]\sqrt{(P-Q)^2 - G^2} \cos 2p \; - \\ &\quad - \frac{\alpha}{8\omega^2}[(P-Q)^2 - G^2] \cos 4p \; - \\ &\quad - \frac{\beta}{4\omega\lambda}Q\sqrt{(P-Q)^2 - G^2}\,[\cos 2(q+2p) + \cos 2q]. \end{aligned} \tag{9}$$

Here q is the slow angle and p is the fast one. The latter can be made cyclic by carrying out a first order Lie transformation $(p', q', P', Q') \to (p, q, P, Q)$, that amounts to averaging (9) over the fast angle [12]. The normalized Hamiltonian becomes

$$\overline{H} = \omega P' + \epsilon \overline{H}_1,$$

where

$$\overline{H}_1(q', P', Q') = \frac{1}{2\pi}\int_0^{2\pi} H_1(p', q', P', Q')\,\mathrm{d}p'. \tag{10}$$

By suppressing primes for brevity, one obtains

$$\begin{aligned} \overline{H} &= \omega P - \epsilon\frac{3\alpha}{8\omega^2}P^2 + \epsilon F(q, Q), \\ F &= \left[\mu + \frac{1}{4}\left(\frac{3\alpha}{\omega^2} - \frac{2\beta}{\omega\lambda}\right)P\right]Q + \frac{1}{8}\left(\frac{4\beta}{\omega\lambda} - \frac{3\alpha}{\omega^2} - \frac{3\gamma}{\lambda^2}\right)Q^2 \; - \\ &\quad - \frac{\beta}{4\omega\lambda}Q\sqrt{(P-Q)^2 - G^2} \cos 2q. \end{aligned} \tag{11}$$

One only has to solve for the motion generated by the Hamiltonian $F(q, Q)$. In the following section, we integrate the relevant equations analytically in terms of Jacobi's elliptic functions.

3. Integration by Quadratures

The Hamiltonian of (11) is of the form

$$F = A_1 Q + A_2 Q^2 + A_3 R(Q) \cos mq, \tag{12}$$

where q and Q are conjugate angle and action variables, respectively, m is an integer and

$$R^2(Q) = B_0 + B_1 Q + B_2 Q^2 + B_3 Q^3 + B_4 Q^4, \tag{13}$$

with A_i's and B_i's being real constants. Such Hamiltonians are integrable in terms of elliptic functions. Relevant equations of motion are

$$\frac{\mathrm{d}q}{\mathrm{d}t} = \frac{\partial F}{\partial Q} = A_1 + 2A_2 Q + A_3 \frac{\mathrm{d}R(Q)}{\mathrm{d}Q} \cos mq, \tag{14a}$$

$$\frac{\mathrm{d}Q}{\mathrm{d}t} = -\frac{\partial F}{\partial q} = m A_3 R(Q) \sin mq. \tag{14b}$$

The Hamiltonian F is an integral of motion. Solving (12) for $\cos mq$ and $\sin mq$, and substituting in (14b) yields

$$\int \frac{\mathrm{d}Q}{\sqrt{f_4(Q)}} = \int m\,\mathrm{d}t, \quad -\infty < t < +\infty, \tag{15}$$

where

$$\begin{aligned} f_4(Q) &= C_0 Q^4 + 4C_1 Q^3 + 6C_2 Q^2 + 4C_3 Q + C_4, \\ C_0 &= A_3^2 B_4 - A_2^2, \quad C_1 = \tfrac{1}{4} A_3^2 B_3 - \tfrac{1}{2} A_1 A_2, \\ C_2 &= \tfrac{1}{6} A_3^2 B_2 + \tfrac{1}{3} F A_2 - \tfrac{1}{6} A_1^2, \\ C_3 &= \tfrac{1}{2} F A_1 + \tfrac{1}{4} A_3^2 B_1, \quad C_4 = A_3^2 B_0 - F^2. \end{aligned} \tag{16}$$

Let z_0 be any root of equation $f_4(Q) = 0$. A change of variable from Q to [13]

$$\begin{aligned} \xi &= D_2 + \frac{D_3}{Q - z_0}, \\ D_2 &= \frac{1}{24} f_4''(z_0), \quad D_3 = \frac{1}{4} f_4'(z_0), \quad (\cdot)' \equiv \frac{\mathrm{d}(\cdot)}{\mathrm{d}Q}, \end{aligned} \tag{17}$$

transforms Equation (15) to

$$\int \frac{\mathrm{d}\xi}{\sqrt{(\xi - \xi_1)(\xi - \xi_2)(\xi - \xi_3)}} = 2m \int \mathrm{d}t, \quad -\infty < t < +\infty, \tag{18}$$

where ξ_1, ξ_2 and ξ_3 are the roots of the following cubic equation

$$4\xi^3 - G_2\xi - G_3 = 0, \tag{19}$$

with

$$G_2 = C_0C_4 - 4C_1C_3 + 3C_2^2,$$

$$G_3 = C_0C_2C_4 + 2C_1C_2C_3 - C_2^3 - C_0C_3^2 - C_1^2C_4.$$

Using the transformation [14]

$$\Lambda^2 = \frac{\xi_1 - \xi_3}{\xi - \xi_3}, \tag{20}$$

Equation (18) can then be written in its canonical form

$$\int \frac{\mathrm{d}\Lambda}{\sqrt{(1-\Lambda^2)(1-\kappa^2\Lambda^2)}} = \chi t + \boldsymbol{c}, \quad -\infty < t < +\infty,$$
$$\chi = m\sqrt{\xi_1 - \xi_3}, \quad \kappa^2 = \frac{\xi_2 - \xi_3}{\xi_1 - \xi_3}. \tag{21}$$

The solution for Λ in terms of Jacobi's elliptic function, sn, is

$$\Lambda = \mathrm{sn}(\chi t + \boldsymbol{c}, \kappa), \tag{22}$$

where $\boldsymbol{c}$ is a constant given by initial conditions. Substituting (22) in (20) and then in (17) gives

$$Q = \frac{E_2\mathrm{sn}^2(\chi t + \boldsymbol{c}, \kappa) - E_1\eta}{\mathrm{sn}^2(\chi t + \boldsymbol{c}, \kappa) - \eta}, \quad \kappa^2 < 1,$$
$$\eta = \frac{\xi_3 - \xi_1}{\xi_3 - D_2}, \quad E_1 = z_0, \quad E_2 = z_0 + \frac{D_3}{\xi_3 - D_2}. \tag{23}$$

Finally from (12) one gets

$$q = \frac{1}{m} \arccos \left[\frac{F - A_1Q - A_2Q^2}{A_3R(Q)}\right]. \tag{24}$$

Our Hamiltonian (11) belongs to the class of (12) with

$$A_1 = \mu + \frac{1}{4}\left(\frac{3\alpha}{\omega^2} - \frac{2\beta}{\omega\lambda}\right)P, \;\; A_2 = \frac{1}{8}\left(\frac{4\beta}{\omega\lambda} - \frac{3\alpha}{\omega^2} - \frac{3\gamma}{\lambda^2}\right), \;\; A_3 = -\frac{\beta}{4\omega\lambda},$$

$$B_2 = P^2 - G^2, \quad B_3 = -2P, \quad B_4 = 1, \quad B_0 = B_1 = 0, \quad m = 2.$$

4. Solutions in Terms of Initial Conditions

From (23) at $t = 0$, $Q = Q_0$ and $q = q_0$, one obtains

$$\begin{aligned}
\mathrm{sn}(\boldsymbol{c}, \kappa) &= \pm\sqrt{\eta \frac{Q_0 - E_1}{Q_0 - E_2}}, \\
\mathrm{cn}(\boldsymbol{c}, \kappa) &= \pm\sqrt{\frac{(1-\eta)Q_0 + (\eta E_1 - E_2)}{Q_0 - E_2}}, \\
\mathrm{dn}(\boldsymbol{c}, \kappa) &= \sqrt{\frac{(1-\kappa^2\eta)Q_0 + (\kappa^2\eta E_1 - E_2)}{Q_0 - E_2}}.
\end{aligned} \tag{25}$$

Therefore, from (12), (16) and (19) one gets

$$\xi_i = \xi_i(q_0, Q_0), \quad (i = 1, 2, 3),$$

$$\eta = \eta(q_0, Q_0), \quad \chi = \chi(q_0, Q_0), \quad \kappa = \kappa(q_0, Q_0).$$

Equations (25) may now be used in the addition theorem for elliptic functions

$$\mathrm{sn}(\chi t + \boldsymbol{c}, \kappa) = \frac{\mathrm{sn}(\chi t, \kappa)\mathrm{cn}(\boldsymbol{c}, \kappa)\mathrm{dn}(\boldsymbol{c}, \kappa) + \mathrm{cn}(\chi t, \kappa)\mathrm{sn}(\boldsymbol{c}, \kappa)\mathrm{dn}(\chi t, \kappa)}{1 - \kappa^2 \mathrm{sn}^2(\chi t, \kappa)\mathrm{sn}^2(\boldsymbol{c}, \kappa)}, \tag{26}$$

to yield

$$\begin{aligned}
V(q_0, Q_0, t) \equiv\ & \mathrm{sn}^2(\chi t + \boldsymbol{c}, \kappa) \\
=\ & \{[(1-\eta)Q_0 + (\eta E_1 - E_2)][(1 - \eta\kappa^2)Q_0\ + \\
& + (\kappa^2\eta E_1 - E_2)] \cdot \mathrm{sn}^2(\chi t, \kappa) + \eta(Q_0 - E_1)(Q_0 - E_2)\ \times \\
& \times \mathrm{cn}^2(\chi t, \kappa) \cdot \mathrm{dn}^2(\chi t, \kappa) \pm 2\mathrm{sn}(\chi t, \kappa) \cdot \mathrm{cn}(\chi t, \kappa) \times \\
& \times \mathrm{dn}(\chi t, \kappa) \cdot [\eta(Q_0 - E_1)(Q_0 - E_2)((1-\eta)Q_0\ + \\
& + (\eta E_1 - E_2))((1 - \eta\kappa^2)Q_0 + (\kappa^2\eta E_1 - E_2))]^{1/2}\} \\
& /((Q_0 - E_2)^2 + \kappa^4\eta^2 \cdot \mathrm{sn}^4(\chi t, \kappa) \cdot (Q_0 - E_1)^2\ - \\
& - 2\kappa^2\eta(Q_0 - E_1)(Q_0 - E_2) \cdot \mathrm{sn}^2(\chi t, \kappa)).
\end{aligned} \tag{27}$$

Thus, Q and q may finally be given in terms of Q_0, q_0 and t as

$$Q(q_0, Q_0, t) = \frac{E_2 V(q_0, Q_0, t) - E_1\eta}{V(q_0, Q_0, t) - \eta}, \tag{28a}$$

$$q(q_0, Q_0, t) = \frac{1}{m} \arccos\left[\frac{A_1(Q_0 - Q) + A_2(Q_0^2 - Q^2) + A_3 R(Q_0)\cos mq_0}{A_3 R(Q)}\right], \tag{28b}$$

which are the closed form solutions of the normalized system. In (11), the quantity under the square root sign should be positive. Thus, $0 \leqslant Q \leqslant P - G$ is the allowable region of the phase space for the motion.

Caranicolas [3] has considered

$$\overline{H} = \overline{H_0} + \overline{H}_1(Q, q) = d_1 Q(d_2 - Q)(\tfrac{1}{2} + \cos 2q), \quad d_1, d_2 = \text{constant}. \tag{29}$$

This is the averaged Hamiltonian of a two-dimensional perturbed harmonic oscillator. Equation (29) is a special case of (12). Thus, his regular solutions may be obtained as a special case of (28).

5. Periodic Orbits

The fixed points of (14) can represent periodic motions. These points are found by solving

$$A_1 + 2A_2 Q + A_3 \frac{\mathrm{d}R(Q)}{\mathrm{d}(Q)} \cos 2q = 0, \tag{30}$$

for Q with $q = 0, \pi/2, \pi, 3\pi/2$. Denote them by (q_j, Q_j). Correspondingly let $L_j = P - Q_j$, $K_j = Q_j$ and $k_j = q_j + l$. For the motion associated with the jth fixed point, one readily finds from (7)

$$\frac{\mathrm{d}g}{\mathrm{d}l} = \frac{\mathrm{d}g}{\mathrm{d}t} \bigg/ \frac{\mathrm{d}l}{\mathrm{d}t} = \frac{\tilde{\Omega} + \epsilon f_1(l)}{\tilde{\omega} + \epsilon f_2(l)} \cong \frac{\tilde{\Omega}}{\tilde{\omega}} + \epsilon \frac{1}{\tilde{\omega}} \left[f_1(l) - \frac{\tilde{\Omega}}{\tilde{\omega}} f_2(l) \right], \tag{31}$$

with

$$\tilde{\Omega} = \Omega + \epsilon \left(\frac{\alpha}{4\omega^2} G + \frac{\beta}{4\omega\lambda} \frac{K_j G}{\sqrt{L_j^2 - G^2}} \cos 2q_j \right), \tag{32a}$$

$$\tilde{\omega} = \omega - \epsilon \left(\frac{3\alpha}{4\omega^2} L_j + \frac{\beta}{2\omega\lambda} K_j + \frac{\beta}{4\omega\lambda} \frac{K_j L_j}{\sqrt{L_j^2 - G^2}} \cos 2q_j \right), \tag{32b}$$

$$f_1(l) = \frac{\partial H_1^{(l)}(L_j, K_j, k_j)}{\partial G}, \qquad f_2(l) = \frac{\partial H_1^{(l)}(L_j, K_j, k_j)}{\partial L}, \tag{32c}$$

where $H_1^{(l)}$ is the collection of those terms of (7b) that contain cosines of $2k, 4k, 2l, 4l$ and $(2k + 2l)$. These are the terms that in the integration of $\mathrm{d}g/\mathrm{d}l$

don't contribute to secular terms in l. Thus,

$$H_1^{(l)} = \frac{1}{2\lambda}\left(\frac{\beta}{\omega}L + \frac{\gamma}{\lambda}K\right)K\cos 2k - \frac{\gamma}{8\lambda^2}K^2\cos 4k + \\ + \frac{1}{2\omega}\left(\frac{\alpha}{\omega}L + \frac{\beta}{\lambda}K\right)\sqrt{L^2-G^2}\cos 2l - \frac{\alpha}{8\omega^2}(L^2-G^2)\cos 4l - \\ - \frac{\beta}{4\omega\lambda}K\sqrt{L^2-G^2}\cos(2k+2l). \quad (33)$$

Integrating (31) gives

$$g = \frac{\tilde{\Omega}}{\tilde{\omega}}l + \epsilon\frac{1}{\tilde{\omega}}\left[g_1(l) - \frac{\tilde{\Omega}}{\tilde{\omega}}g_2(l)\right], \quad (34)$$

where

$$g_1(l) = \int_0^l f_1(u)\,\mathrm{d}u, \qquad g_2(l) = \int_0^l f_2(u)\,\mathrm{d}u.$$

The functions $g_1(l)$ and $g_2(l)$ are 2π-periodic in l. Thus, according to (4), (5) and (34), the system undergoes periodic motion in l if $\tilde{\Omega}/\tilde{\omega}$ is rational. In order to visualize the shapes of the orbits, we use polar (r, θ, z) coordinates. Any orbit associated with the fixed points of (14) lies on a surface of revolution with the parametric equations

$$\sum : \begin{cases} r = \sqrt{\dfrac{1}{\omega}[L_j - (L_j^2 - G^2)^{1/2}\cos 2l]}, \\ \theta \in [0, 2\pi[, \\ z = \left(\dfrac{2K_j}{\lambda}\right)^{1/2}\sin(q_j + l), \\ l \in [0, 2\pi[. \end{cases} \quad (35)$$

Depending on the values of q_j's, two general types of periodic and quasi-periodic orbits emerge. Type I corresponds to $q_j = 0, \pi$ for which Σ is a hyperboloid of one sheet, Figure 1(a). Type II is for $q_j = \pi/2, 3\pi/2$. In this case, Σ is a spheroidal surface, Figure 1(b), conformant with the shape of the galaxy. Neither the hyperboloid nor the spheroid are complete. They are confined to surfaces $z = \pm(2K_j/\lambda)^{1/2}$. For irrational $\tilde{\Omega}/\tilde{\omega}$, the orbits will be quasi-periodic and dense on Σ. Such classification of orbits is not altered if one continues the normalization of the Hamiltonian up to the second order in ϵ. The origin of the two-dimensional phase space of the normalized system is the trivial fixed point associated with equatorial motions. The nature of this point plays an important role in the stability of oblate and prolate galaxies which is the subject of the next section.

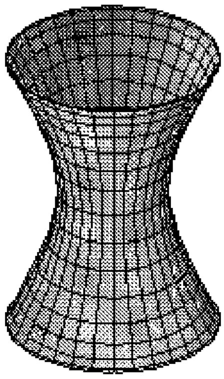

Figure 1a

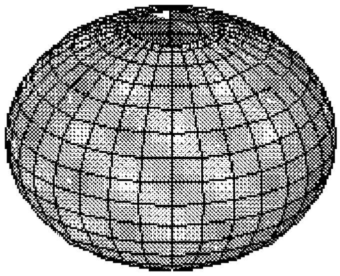

Figure 1b

6. Local Stability

In order to investigate the stability of equatorial motions, we introduce a canonical change of variables as follows

$$Q = \frac{1}{2}(\xi^2 + \eta^2), \quad q = \arctan \frac{\xi}{\eta}. \tag{36}$$

Equations (14) transform to

$$\dot{\xi} = A_1\eta + A_2\eta(\xi^2 + \eta^2) + A_3\eta R_1(\xi, \eta) + \tfrac{1}{2}A_3\eta R_2(\xi, \eta), \tag{37a}$$

$$\dot{\eta} = -A_1\xi - A_2\xi(\xi^2 + \eta^2) + A_3\xi R_1(\xi, \eta) + \tfrac{1}{2}A_3\xi R_2(\xi, \eta), \tag{37b}$$

with

$$R_1(\xi, \eta) = \left[(P^2 - \tfrac{1}{2}\xi^2 - \tfrac{1}{2}\eta^2)^2 - G^2\right]^{1/2},$$

$$R_2(\xi, \eta) = \frac{1}{R_1(\xi, \eta)}(\eta^2 - \xi^2)\left(P - \frac{1}{2}\xi^2 - \frac{1}{2}\eta^2\right).$$

Linearizing (37) about the origin, ($\xi = \eta = 0$), gives

$$\dot{\xi} = \left(A_1 + A_3\sqrt{P^2 - G^2}\right)\eta, \tag{38a}$$

$$\dot{\eta} = \left(-A_1 + A_3\sqrt{P^2 - G^2}\right)\xi. \tag{38b}$$

Correspondingly, the characteristic equation becomes

$$w^2 - \Delta = 0, \quad \Delta = A_3^2(P^2 - G^2) - A_1^2. \tag{39}$$

The origin is a center if $\Delta < 0$ (stable equatorial motions) and a saddle if $\Delta > 0$ (unstable equatorial motions). By combining these conditions with $G^2 < P^2$ (this is a direct result of (6) and (8)), the following conditions are obtained

$$f(P,e) > 0 \Rightarrow \begin{cases} \Delta > 0 & \text{if} \quad G^2 < f(P,e) < P^2, \\ \Delta < 0 & \text{if} \quad f(P,e) < G^2 < P^2, \end{cases} \tag{40a}$$

$$f(P,e) < 0 \Rightarrow \quad \Delta < 0, \tag{40b}$$

where

$$f(P,e) = (c_2 P^2 - 2c_1 P - c_{01})/c_{02},$$
$$c_{01} = 16\mu^2\omega^4\lambda^4, \quad c_{02} = \beta^2\omega^2\lambda^2, \quad c_1 = 4\mu(3\alpha\omega^2\lambda^4 - 2\beta\omega^3\lambda^3),$$
$$c_2 = 12\alpha\beta\omega\lambda^3 - 9\alpha^2\lambda^4 - 3\beta^2\omega^2\lambda^2.$$

The analytical results of this paper are reliable for $P < 1$ and $0 < e < 0.6$. Outside this range of the parameters, perturbation approximation and closeness to 1:1 resonance break down. We have calculated $f(P,e)$ in this range of validity and plotted in Figures 2(a) and 2(b) for prolate and oblate galaxies, respectively. The origin of the averaged system can become a saddle point in prolate galaxies (Figure 2(a)) indicating unstable equatorial motions. In oblate galaxies, however, the origin remains a center leading to stable equatorial motions.

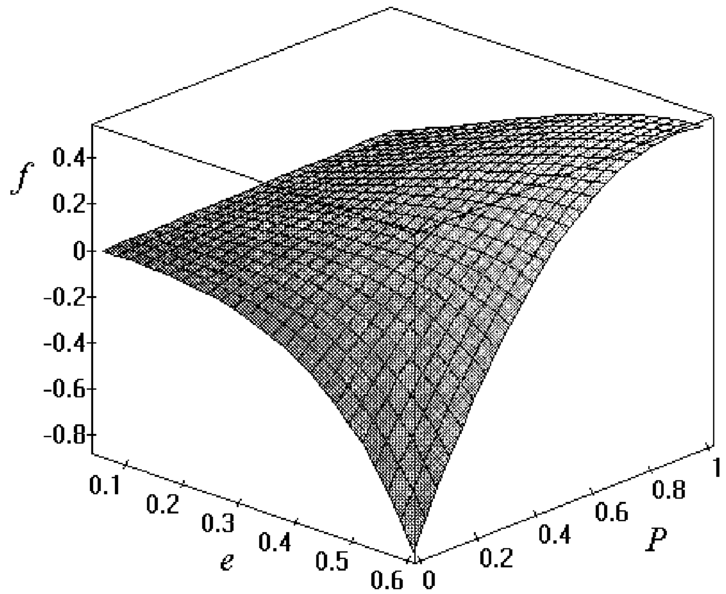

Figure 2a. Variations of $f(P,e)$ in prolate galaxies.

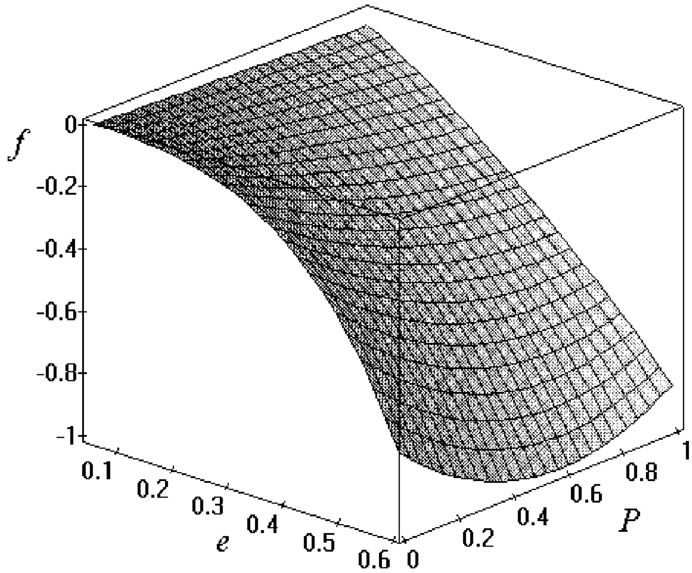

Figure 2b. Variations of $f(P, e)$ in oblate galaxies.

7. Examples

We have used the formalism developed in this paper to study the orbits for $e^2 = 0.19$. The constant parameters ω, λ, α, β, and γ are calculated from (2) and are given in Table I. For motions with $P = 0.6$ and for several choices of G and initial conditions, we have calculated $\overline{Z} = (2Q)^{1/2}\cos q$ and $\overline{z} = (2Q)^{1/2}\sin q$ and plotted them in Figures 3 and 4. All flows are confined to circles of radius $R_0 = (2P - 2G)^{1/2}$. Center-type fixed points exist in both oblate and prolate spheroids. They indicate stable periodic or quasi-periodic solutions. The origin itself stands for equatorial motions. In the prolate system and for certain values of the angular momentum, the origin is a saddle point. This implies that the equatorial motions of the prolate spheroid are unstable and are subject to appreciable levitation in the z-direction. Periodic and quasi-periodic orbits of Type II and their surrounding tubes, exist in both the oblate and prolate systems. Orbits of Type I are to be found only in the prolate spheroid.

TABLE I

Numerical values for the constant coefficients appearing in the original Hamiltonian for $e^2 = 0.19$.

e^2	Shape	ω	λ	α	β	γ
0.19	Oblate	0.242334	0.258052	0.006776	0.007865	0.009147
0.19	Prolate	0.266440	0.250022	0.009872	0.008480	0.007298

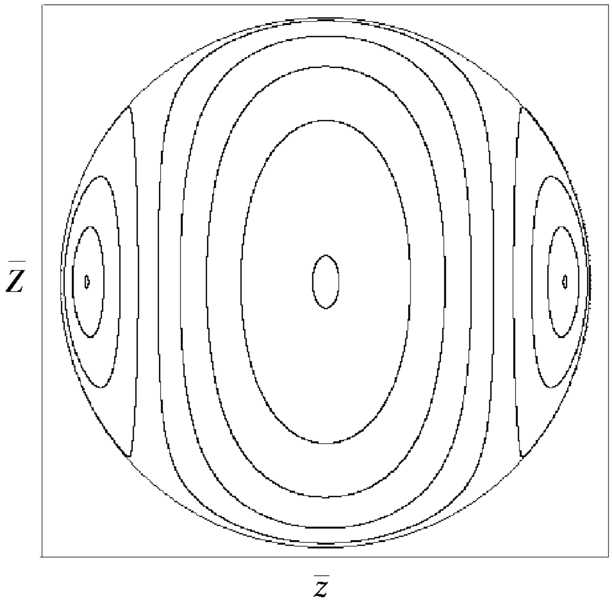

Figure 3a. The phase portrait of the normalized Hamiltonian for an oblate spheroid with $e^2 = 0.19$, $P = 0.6$ and $G = 0.2$.

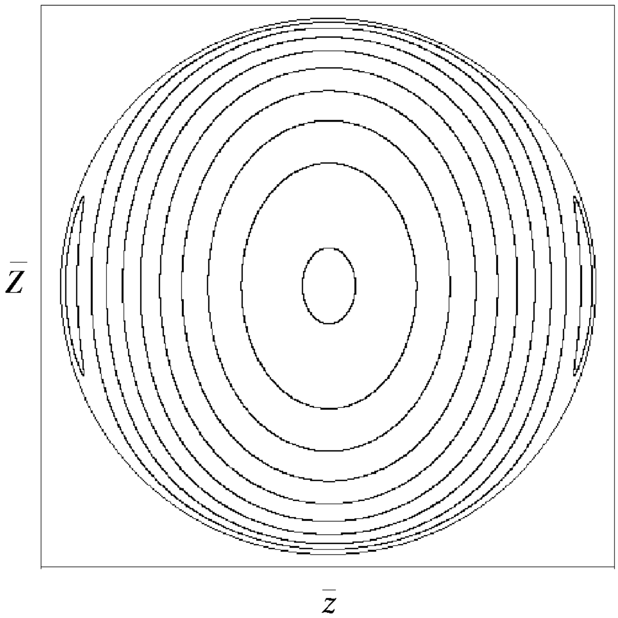

Figure 3b. The phase portrait of the normalized Hamiltonian for an oblate spheroid with $e^2 = 0.19$, $P = 0.6$ and $G = 0.5$.

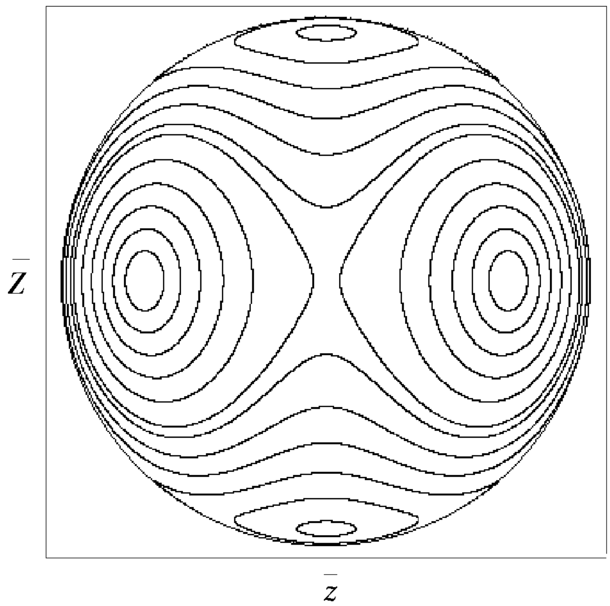

Figure 4a. The phase portrait of the normalized Hamiltonian for a prolate spheroid with $e^2 = 0.19$, $P = 0.6$ and $G = 0.2$.

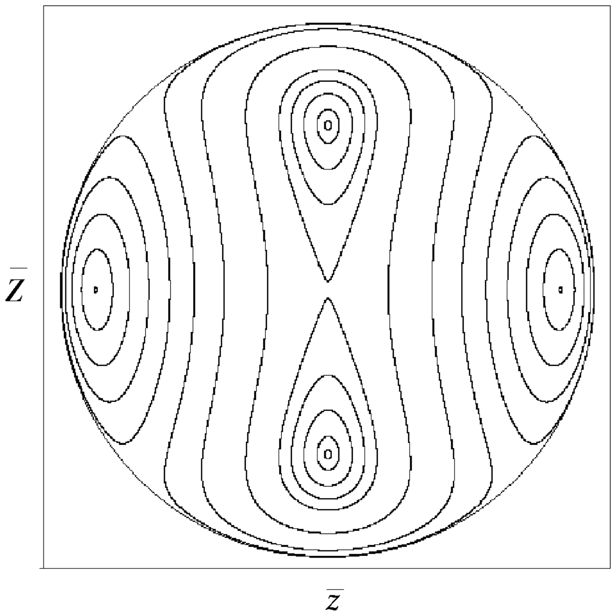

Figure 4b. The phase portrait of the normalized Hamiltonian for a prolate spheroid with $e^2 = 0.19$, $P = 0.6$ and $G = 0.5$.

8. Discussion

This research was motivated by the work of Davoust [8]. The model presents motions in spheroidal galaxies with quadratic density distributions:

(a) By the Lissajous transformation of Deprit [11] and a transformation to action angle variables, the Hamiltonian is written in a form amenable to standard perturbation techniques.
(b) By a transformation to a rotating coordinate system the Hamiltonian is split into terms containing slowly and rapidly varying angles.
(c) The fast angle is averaged out by a Lie transformation and a normalized Hamiltonian with one degree of freedom is obtained.
(d) Finally, after some algebraic manipulations, the normalized system is integrated in terms of Jacobi's elliptic functions. Jacobi's functions are doubly periodic and have most of the useful properties of trigonometric functions such as addition rules, the double angle formulae, etc. These features permit one to express the solutions explicitly in terms of the initial conditions and time.

Davoust [8] studied the existence of periodic orbits at exact resonances for rational values of ω/λ. The formalism adopted in this paper allows us to consider irrational ω/λ as well.

Chaotic motions theoretically occur in the vicinity of the saddle points of the averaged system. Thus, according to the results of Section 6, chaotic orbits can emerge in prolate galaxies. On the evidence of Figure (2), oblate galaxies are more stable than prolate ones, an intuitively expected feature, for, prolate galaxies are exceptional occurrences.

Similarities exist between the findings of this paper and those of separable models. According to Stäckel's theorem, the Hamilton–Jacobi equation is separated in elliptical coordinates for Hamiltonians of the form (de Zeeuw, [15])

$$\begin{aligned} H &= \tfrac{1}{2}(f_u^{-2} p_u^2 + f_v^{-2} p_v^2 + f_w^{-2} p_w^2) + \Phi(u, v, w), \\ \Phi(u, v, w) &= -\frac{f(u)}{(u-v)(u-w)} - \frac{f(v)}{(v-w)(v-u)} - \frac{f(w)}{(w-u)(w-v)}, \end{aligned}$$

where f is an arbitrary function and f_u^{-2}, f_v^{-2} and f_w^{-2} are functions of u, v and w (see also [16]). In such systems four types of orbits emerge: *boxes, short axis tubes, outer long axis tubes* and *inner long axis tubes*. Although the system (1) is not separable, it has similar types of orbits. Inner and outer long axis tubes of de Zeeuw are similar to our Type I and Type II orbits of prolate spheroids, respectively. Short axis tubes of de Zeeuw are the analogs of our Type II orbits for oblate spheroids. Closed curves in the phase portrait of the averaged system (Figures 3 and 4) which encircle the origin, are associated with box type orbits as seen in the *x*-*z* and *y*-*z* planes. However, they form tubes in the *x*-*y* plane, for G is constant and does not reverse sign during the motion.

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