

FLUORESCENT SCATTERING IN PLANETARY ATMOSPHERES

II. COUPLING AMONG TRANSITIONS*

YOUSUF SOBOUTI

Yerkes Observatory

Received October 27, 1961

ABSTRACT

The ground energy level of an atmospheric constituent absorbs the solar radiation in those frequencies which arise from this level. Re-emission may take place to the original level (resonant transition) or some other levels (fluorescent transitions). When more than one lower level is populated, an upper level may become populated through any one of several absorption transitions. Scattered radiation in one line may be absorbed again and then re-emitted in another line. The radiative-transfer equation for each line must therefore involve all transitions capable of populating the same upper level; the intensities of different emissions having a common upper level will be "coupled" to one another.

In Part A the coupled transfer equations appropriate to a planetary atmosphere are developed and solved for diffusely reflected and transmitted radiations. In both optically finite and semi-infinite atmospheres the zeroth and first iterates of the integral equations have been obtained in terms of Chandrasekhar's X -, Y -, and H -functions. These iterates are sufficient for analytic solutions in the case of nearly symmetric coupling between every pair of transitions. Part B treats the problem of coupling when continuous absorption exists across the line profiles. The principle of reciprocity is extended to the problem of coupled fluorescent scattering.

A. FLUORESCENT SCATTERING WITHOUT CONTINUOUS ABSORPTION

I. INTRODUCTION

Let us consider the atmospheric molecules in a given vibrational level v'' , illuminated by sunlight. Upon absorbing solar radiation, these molecules will make a transition to some higher level, v' . Once an upper level is populated, downward transitions from it will emit all bands in the corresponding v'' -progression (i.e., the progression with $v' \equiv$ constant). Thus, by this *fluorescent mechanism*, the solar radiation absorbed in a given frequency may be re-emitted in different frequencies. Although not necessarily isotropic, re-emission of fluorescent photons may take place in any direction, and isotropic scattering is frequently a good approximation. We refer to this process as *fluorescent scattering*.

Absorption of direct solar radiation and scattered photons will occur only in those lines or bands whose lower levels are populated (see Fig. 2 of Paper I). If, however, more than one v'' -level is populated, there will be a coupling among the scattered intensities of different bands. A particular fluorescent band will arise from the absorption in more than one band.

At temperatures of the order of 300° K, typical of the upper atmosphere of the earth, most of the atmospheric constituents are in their ground vibrational levels. Conceivably, however, more than one vibrational level of the lowest electronic state could have an appreciable, though small, population. A more important manifestation of coupling occurs in the rotational structure. The populated rotational levels, within the ground vibrational level, will allow an upper rotational level to be populated by absorptions in two or more lines.

The content of this paper has grown out of an attempt to find how the relative intensities of molecular lines and bands, produced in planetary atmospheres by fluorescent scattering of solar radiation, are affected by vibrational and (especially) rotational coupling. Although the equations are developed and solved for the specific (or mono-

* The research reported in this paper was supported in part by the National Aeronautics and Space Administration through research grant NsG 118-61.

chromatic) intensity, $I^{(\nu)}$, integration over a line profile may be performed with the techniques treated in Paper I (Chamberlain and Sobouti 1962).

In the following sections we have developed the radiative-transfer equation appropriate to this problem. The intensities emerging from the top and bottom of the atmosphere are expressed by means of *scattering* and *transmission matrices* whose elements $S_{ij}(\mu, \mu_0)$ and $T_{ij}(\mu, \mu_0)$ involve the two transitions i and j . This technique was first applied by Busbridge and Stibbs (1954; also see Busbridge 1955) to coupled multiplet lines in semi-infinite atmospheres. Their solution may be obtained as a special case of that given here, when the atmosphere is semi-infinite (see Sec. A, IV).

We shall express the solutions for the emergent intensities in terms of Chandrasekhar's X -, Y -, and H -functions by a method of iteration. Only zeroth and first iterates have been found, which suffice, however, to solve the problem in the case of nearly symmetric coupling—i.e., when the coupling of line i to line j is nearly equal to the coupling of j to i . The problem of coupling among frequencies treated here is quite general. It could include the coupling among bands or lines or both, and the letters J' and J'' in the following sections are not meant to represent rotational levels exclusively.

II. EQUATIONS OF FLUORESCENT SCATTERING

Let $I_{J'J''}^{(\nu)}$ denote the specific intensity of the radiation field in a transition $J'-J''$. The superscript (ν) represents the frequency dependence of the intensity within the line profile. The vertical optical thickness of an atmosphere is defined by

$$t_{J'J''}^{(\nu)} = \int_z^\infty \kappa_{J'J''}^{(\nu)}(z') dz', \tag{1}$$

where $\kappa_{J'J''}^{(\nu)}$ is the volume absorption coefficient and z is the altitude.

In the present problem, however, we shall find it convenient to express this optical depth as

$$dt_{J'J''}^{(\nu)} = k_{J'J''}^{(\nu)} dt^{(\nu)}, \tag{2}$$

where

$$k_{J'J''}^{(\nu)} = \frac{\kappa_{J'J''}^{(\nu)}}{\sum_{J''} \kappa_{J'J''}^{(\nu)}} \tag{3}$$

and

$$dt^{(\nu)} = - \sum_{J''} \kappa_{J'J''}^{(\nu)} dz. \tag{4}$$

The profiles of all lines are here considered to be similar. The variable $t^{(\nu)}$ is then the sum of the optical thicknesses at corresponding frequencies in all the lines participating in absorption to level J' . It is independent of any particular value of J'' and therefore serves as a convenient standard for optical depth (although it is without physical significance). We assume that the atmosphere has a constant temperature everywhere; $\kappa_{J'J''}^{(\nu)}$ will then be independent of altitude and therefore of the standard optical thickness $t^{(\nu)}$.

A description of the albedo for single scattering, ϖ , is given in Paper I (Chamberlain and Sobouti 1962). In the case of isotropic scattering with no continuous absorption, with which we shall be concerned, the albedos are

$$\varpi_{J'J''} = \frac{A_{J'J''}}{\sum_{J''} A_{J'J''}}, \tag{5}$$

where $A_{J'J''}$ is Einstein's coefficient of spontaneous emission for a transition $J' \rightarrow J''$. More specifically for a molecular (vibrational-rotational) transition, v' ; $J' \rightarrow v''$; J'' , equation (5) becomes

$$\begin{aligned} \bar{\omega}_{J'J''} &= \frac{A_{v'v''}}{\sum_{v''} A_{v'v''}} \frac{A_{J'J''}}{\sum_{J''} A_{J'J''}} \\ &= \frac{A_{v'v''}}{\sum_{v''} A_{v'v''}} \frac{S(J'J'')}{(2J'+1)}, \end{aligned} \quad (6)$$

where $S(J'J'')$ is the line strength of the transition $J' \rightleftharpoons J''$ (see Herzberg 1950, p. 127). Hereafter, for the sake of brevity, we shall omit the indices J' and v and replace J'' by i .

For isotropic, resonant scattering the equation of transfer is (Chandrasekhar 1950, p. 22; hereafter this work will be called simply "R.T.")

$$\mu \frac{dI_i(t, \mu)}{dt} = k_i I_i(t, \mu) - \frac{1}{2} \bar{\omega}_i k_i \int_{-1}^1 I_i(t, \mu') d\mu' - \frac{1}{4} \bar{\omega}_i k_i F_i e^{-k_i t / \mu_0}, \quad (7)$$

where $\mu = \cos \theta$, and θ is the angle between the upward normal to the plane of stratification of the atmosphere and the direction to which I_i refers. The left-hand side of equation (7) is the total rate of change of I_i . The first term on the right-hand side is the loss rate due to absorption. The second term is the gain due to re-emission of the absorbed radiation, which is only a fraction $\bar{\omega}_i$ of the latter. Finally, the last term is the contribution from the external flux πF_i incident from the direction μ_0 , and again involves the fraction $\bar{\omega}_i$. The factor $\exp(-k_i t / \mu_0)$ takes account of the attenuation of the incident flux after it has traversed an optical path $k_i t / \mu_0$.

The fraction $(1 - \bar{\omega}_i)$ of the absorbed radiation which is lost to the intensity I_i will appear in other transitions. In like manner, if absorption to the particular upper level in question occurs in several possible transitions, the i th emission will receive some contribution from the absorbed intensities in other transitions. For instance, the contribution of absorption in the j th transition to the i th emission will be

$$- \frac{1}{2} \bar{\omega}_i k_j \int_{-1}^1 I_j(t, \mu') d\mu' - \frac{1}{4} \bar{\omega}_i k_j F_j e^{-k_j t / \mu_0}.$$

When such contributions from all transitions are taken into account, the complete equation of transfer for the i th emission becomes

$$\begin{aligned} \mu \frac{dI_i(t, \mu)}{dt} &= k_i I_i(t, \mu) - \frac{\bar{\omega}_i}{2} \sum_j^n k_j \int_{-1}^1 I_j(t, \mu') d\mu' \\ &\quad - \frac{\bar{\omega}_i}{4} \sum_j^n k_j F_j e^{-k_j t / \mu_0}, \quad i = 1, 2, \dots, n, \end{aligned} \quad (8)$$

where n is the number of bands of which the lower levels are populated. This equation does not include any processes that populate J' through absorption to higher levels and subsequent cascade.

Equations (8) are a set of n equations coupled to one another through the terms under the summation signs explicitly and albedos implicitly. The higher the value of $\bar{\omega}_i k_j$, the stronger is the coupling of i to j .

III. SOLUTION OF THE COUPLED TRANSFER EQUATIONS: FINITE ATMOSPHERES

a) Principles of Invariance: The Integral Equations

Let τ denote the standard optical thickness of the atmosphere as defined by equation (4). We are primarily interested in solutions for diffuse radiations emerging from the top ($t = 0$) and the bottom ($t = \tau$) of the atmosphere, i.e., $I_i(0, +\mu)$ and $I_i(\tau, -\mu)$. (Hereafter μ , as is traditional, will represent $|\cos \theta|$ only.)

Chandrasekhar's method of employing certain "principles of invariance" is adopted with some generalizations appropriate to coupled fluorescence to solve equations (8). The linearity of the inhomogeneous terms involving F_j implies that the solutions must consist of n linear terms, each corresponding to one F_j . Thus let us postulate that the diffuse reflection and transmission at the top and bottom of the atmosphere are given as follows:

$$I_i(0, +\mu) = \sum_j I_{ij}(0, +\mu) = \frac{1}{4\mu} \sum_j F_j S_{ij}(\tau | \mu, \mu_0), \tag{9}$$

$$I_i(\tau, -\mu) = \sum_j I_{ij}(\tau, -\mu) = \frac{1}{4\mu} \sum_j F_j T_{ij}(\tau | \mu, \mu_0). \tag{10}$$

In equations (9) and (10), I_{ij} is the contribution to the emission in frequency i arising from scattering of the incident flux in frequency j . Here S_{ij} and T_{ij} , which are functions of two frequencies i and j , express the manner in which an external flux πF_j , incident from direction μ_0 , contributes to $I_i(0, +\mu)$ and $I_i(\tau, -\mu)$, respectively. The S_{ij} and T_{ij} 's may be considered as the elements of two square matrices, whose diagonal terms, S_{ii} and T_{ii} , are Chandrasekhar's scattering and transmission function (*R.T.*, pp. 20 ff.). We will, however, extend his terminology to include off-diagonal terms as well.

The method of finding the latter functions follows the line of argument that Chandrasekhar employed to derive his single-frequenced functions. Here we shall only outline the method of derivation, rather than give the details of the calculation. We quote the following four principles of Chandrasekhar (*R.T.*, pp. 162 ff.) and extend them to fluorescence.

i) *The intensity $I_i(t, +\mu)$ in the outward direction at any level t results from the reflection of the reduced incident fluxes, $\pi F_j \exp(-k_j t / \mu_0)$, and the diffuse radiations, $I_j(t, -\mu')$, ($0 < \mu' \leq 1$) incident on the surface t by the atmosphere of optical thickness $(\tau - t)$ below t . That is,*

$$I_i(t, +\mu) = \frac{1}{4\mu} \sum_j F_j e^{-k_j t / \mu_0} S_{ij}(\tau - t | \mu, \mu_0) + \frac{1}{2\mu} \sum_j \int_0^1 I_j(t, -\mu') S_{ij}(\tau - t | \mu, \mu') d\mu'. \tag{11}$$

ii) *The intensity $I_i(t, -\mu)$ in the inward direction at any level t results from the transmission of the incident fluxes by the atmosphere of optical thickness t , above the surface t , and the reflection by this same surface of the diffuse radiations $I_j(t, +\mu')$ ($0 \leq \mu' < 1$) incident on it from below. That is,*

$$I_i(t, -\mu) = \frac{1}{4\mu} \sum_j F_j T_{ij}(t | \mu, \mu_0) + \frac{1}{2\mu} \sum_j \int_0^1 I_j(t, +\mu') S_{ij}(t | \mu, \mu') d\mu'. \tag{12}$$

iii) *The diffuse reflection of the incident light by the entire atmosphere is equivalent to the reflection by the part of the atmosphere of optical thickness t , above the level t , and the trans-*

mission by this same atmosphere of the diffuse radiations $I_j(t, +\mu')$ ($0 \leq \mu' \leq 1$) incident on the surface t from below. That is,

$$\frac{1}{4\mu} \sum_j F_j S_{ij}(\tau | \mu, \mu_0) = \frac{1}{4\mu} \sum_j F_j S_{ij}(t | \mu, \mu_0) + e^{-k_i t / \mu} I_i(t, +\mu) + \frac{1}{2\mu} \sum_j \int_0^1 I_j(t, +\mu') T_{ij}(t | \mu, \mu') d\mu'. \quad (13)$$

iv) The diffuse transmission of the incident light by the entire atmosphere is equivalent to the transmission of the reduced incident fluxes $\pi F_j \exp(-k_j t / \mu_0)$ and the diffuse radiations $I_j(t, -\mu')$ ($0 \leq \mu' \leq 1$) incident on the surface t by the atmosphere of optical thickness $(\tau - t)$ below t . That is,

$$\frac{1}{4\mu} \sum_j F_j T_{ij}(\tau | \mu, \mu_0) = \frac{1}{4\mu} \sum_j F_j e^{-k_j t / \mu_0} T_{ij}(\tau - t | \mu, \mu_0) + e^{-k_i(\tau - t) / \mu} I_i(t, -\mu) + \frac{1}{2\mu} \sum_j \int_0^1 I_j(t, -\mu') T_{ij}(\tau - t | \mu, \mu') d\mu'. \quad (14)$$

Equations (11)–(14) suffice to solve for the scattering and transmission functions and their derivatives with respect to τ . The process of solution consists of the following steps:

- Differentiating equations (11)–(14) with respect to t , then passing to the limit $t = 0$ in equations (11) and (13) and to the limit $t = \tau$ in equations (12) and (14).
- Substituting for the derivatives dI_j/dt (evaluated at $t = 0$ or $t = \tau$) from the transfer equation (8).
- Expressing all $I_i(0$ or $\tau, \pm\mu)$ in terms of S_{ij} and T_{ij} by means of equations (9) and (10).
- Equating coefficients of each F_j separately.

Completing these steps, one gets four equations in S_{ij} , T_{ij} , and their derivatives:

$$\left(\frac{k_i}{\mu} + \frac{k_j}{\mu_0}\right) S_{ij}(\tau | \mu, \mu_0) = \varpi_i k_j [\mathfrak{X}_i(\mu) \tilde{\mathfrak{X}}_j(\mu_0) - \mathfrak{Y}_i(\mu) \tilde{\mathfrak{Y}}_j(\mu_0)], \quad (15)$$

$$\left(\frac{k_i}{\mu} - \frac{k_j}{\mu_0}\right) T_{ij}(\tau | \mu, \mu_0) = \varpi_i k_j [\mathfrak{X}_i(\mu) \tilde{\mathfrak{Y}}_j(\mu_0) - \mathfrak{Y}_i(\mu) \tilde{\mathfrak{X}}_j(\mu_0)], \quad (16)$$

$$\frac{\partial S_{ij}(\tau | \mu, \mu_0)}{\partial \tau} = \varpi_i k_j \mathfrak{Y}_i(\mu) \tilde{\mathfrak{Y}}_j(\mu_0), \quad (17)$$

and

$$\left(\frac{k_j}{\mu_0} - \frac{k_i}{\mu}\right) \frac{\partial T_{ij}(\tau | \mu, \mu_0)}{\partial \tau} = \varpi_i k_j \left[\frac{k_j}{\mu_0} \mathfrak{X}_i(\mu) \tilde{\mathfrak{Y}}_j(\mu_0) - \frac{k_i}{\mu} \mathfrak{Y}_i(\mu) \tilde{\mathfrak{X}}_j(\mu_0) \right], \quad (18)$$

where the four \mathfrak{X} - and \mathfrak{Y} -functions are defined as

$$\mathfrak{X}_i(\mu) = 1 + \frac{1}{2} \sum_l \frac{\varpi_l}{\varpi_i} \int_0^1 S_{li}(\tau | \mu, \mu') \frac{d\mu'}{\mu'}, \quad (19)$$

$$\tilde{\mathfrak{X}}_i(\mu) = 1 + \frac{1}{2} \sum_l \frac{k_l}{k_i} \int_0^1 S_{li}(\tau | \mu', \mu) \frac{d\mu'}{\mu'}, \quad (20)$$

$$\mathfrak{Y}_i(\mu) = e^{-k_i\tau/\mu} + \frac{1}{2} \sum_l \frac{\varpi_l}{\varpi_i} \int_0^1 T_{il}(\tau | \mu, \mu') \frac{d\mu'}{\mu'}, \tag{21}$$

and

$$\tilde{\mathfrak{Y}}_i(\mu) = e^{-k_i\tau/\mu} + \frac{1}{2} \sum_l \frac{k_l}{k_i} \int_0^1 T_{li}(\tau | \mu', \mu) \frac{d\mu'}{\mu'}. \tag{22}$$

Expressing the integrands in the last four equations by means of equations (15) and (16), we obtain the four integral equations,

$$\mathfrak{X}_i(\mu) = 1 + \frac{\mu}{2} \sum_l \varpi_l k_l \int_0^1 \frac{d\mu'}{k_l\mu + k_i\mu'} [\mathfrak{X}_i(\mu) \tilde{\mathfrak{X}}_l(\mu') - \mathfrak{Y}_i(\mu) \tilde{\mathfrak{Y}}_l(\mu')], \tag{23}$$

$$\tilde{\mathfrak{X}}_i(\mu) = 1 + \frac{\mu}{2} \sum_l \varpi_l k_l \int_0^1 \frac{d\mu'}{k_l\mu + k_i\mu'} [\tilde{\mathfrak{X}}_i(\mu) \mathfrak{X}_l(\mu') - \tilde{\mathfrak{Y}}_i(\mu) \mathfrak{Y}_l(\mu')], \tag{24}$$

$$\mathfrak{Y}_i(\mu) = e^{-k_i\tau/\mu} + \frac{\mu}{2} \sum_l \varpi_l k_l \int_0^1 \frac{d\mu'}{k_l\mu - k_i\mu'} [\mathfrak{Y}_i(\mu) \tilde{\mathfrak{X}}_l(\mu') - \mathfrak{X}_i(\mu) \tilde{\mathfrak{Y}}_l(\mu')], \tag{25}$$

and

$$\tilde{\mathfrak{Y}}_i(\mu) = e^{-k_i\tau/\mu} + \frac{\mu}{2} \sum_l \varpi_l k_l \int_0^1 \frac{d\mu'}{k_l\mu - k_i\mu'} [\tilde{\mathfrak{Y}}_i(\mu) \mathfrak{X}_l(\mu') - \tilde{\mathfrak{X}}_i(\mu) \mathfrak{Y}_l(\mu')]. \tag{26}$$

The distinction between \mathfrak{X} , \mathfrak{Y} and their transposed functions is only apparent. If one wishes to accept on physical grounds that the solutions of these equations are unique, then $\mathfrak{X}(\mu) = \tilde{\mathfrak{X}}(\mu)$ and $\mathfrak{Y}(\mu) = \tilde{\mathfrak{Y}}(\mu)$ are solutions, which may be verified readily. Hereafter we shall not distinguish between \mathfrak{X} , \mathfrak{Y} and their transposed functions.

Corollary 1.—From the defining equations (23)–(25) for the \mathfrak{X} - and \mathfrak{Y} -functions, one may immediately show that

$$\mathfrak{X}_i(k_i\mu) = \mathfrak{X}_j(k_j\mu), \tag{27}$$

and

$$\mathfrak{Y}_i(k_i\mu) = \mathfrak{Y}_j(k_j\mu). \tag{28}$$

These important relations, which can be verified by separate substitution of $k_i\mu$ and $k_j\mu$ for μ in the defining equations of the \mathfrak{X} - and \mathfrak{Y} -functions, were first noted by Busbridge and Stibbs (1954) for their H -function of semi-infinite atmospheres.

Corollary 2.—Using relations (27) and (28) in equations (15) and (16), one gets

$$\frac{S_{il}(k_i\mu, \mu_0)}{\varpi_i} = \frac{S_{jl}(k_j\mu, \mu_0)}{\varpi_j} \tag{29}$$

and

$$\frac{T_{il}(k_i\mu, \mu_0)}{\varpi_i} = \frac{T_{jl}(k_j\mu, \mu_0)}{\varpi_j}. \tag{30}$$

In turn, when these two equations are used in equations (9) and (10), one obtains

$$\frac{k_i}{\varpi_i} I_i(0, +k_i\mu) = \frac{k_j}{\varpi_j} I_j(0, +k_j\mu) \tag{31}$$

and

$$\frac{k_i}{\varpi_i} I_i(\tau, -k_i\mu) = \frac{k_j}{\varpi_j} I_j(\tau, -k_j\mu). \tag{32}$$

Indeed, a more general relation than equations (31) and (32) is

$$\frac{k_i}{\varpi_i} I_i(t, k_i \mu) = \frac{k_j}{\varpi_j} I_j(t, k_j \mu), \tag{33}$$

which can be verified by a direct substitution in the transfer equation (8) (see the proof for eq. [24] of Paper I).

b) Iterative Solutions of the \mathfrak{X} - and \mathfrak{Y} -Functions

It is intuitively clear that if the albedos are all equal ($\varpi_1 = \varpi_2 = \dots = \varpi$) and the absorption coefficients are equal ($k_1 = k_2 = \dots = k$) and if the incident fluxes are identical ($F_1 = F_2 = \dots = F$), the emergent intensity of every line is the same, and the transfer problem is in no way different from what it would be in the complete absence of any coupling. It is also apparent that this situation is unchanged even when the incident fluxes are different, since the scattered light is distributed equally over all lines by the equal albedos, and that we may treat this "symmetric-coupling" problem with an average flux, πF . Mathematically, these conditions lead immediately to the equality of all I_j 's, as equations (8) become

$$\mu \frac{dI(t, \mu)}{dt} = kI(t, \mu) - \frac{n\varpi}{2} k \int_0^1 I(t, \mu') d\mu' - \frac{n\varpi}{2} kF e^{-kt/\mu_0}. \tag{34}$$

This equation is the same as the non-coupled equation (7) with an albedo $n\pi$. Its solution leads to Chandrasekhar's X - and Y -functions (instead of our \mathfrak{X} and \mathfrak{Y}), which satisfy the integral equations (*R.T.*, p. 181 and chap. viii):

$$X(\mu) = 1 + \frac{\mu}{2} \varpi \int_0^1 \frac{d\mu'}{\mu + \mu'} [X(\mu)X(\mu') - Y(\mu)Y(\mu')] \tag{35}$$

and

$$Y(\mu) = e^{-k\tau/\mu} + \frac{\mu}{2} \varpi \int_0^1 \frac{d\mu'}{\mu - \mu'} [Y(\mu)X(\mu') - X(\mu)Y(\mu')]. \tag{36}$$

When the conditions of symmetric-coupling (i.e., the equality of the ϖ_j 's and k_j 's) are approached, we expect \mathfrak{X} and \mathfrak{Y} to approach X - and Y -functions. Thus, for the zeroth iterates of \mathfrak{X} and \mathfrak{Y} , we may assume

$$\begin{aligned} \mathfrak{X}_i^{(0)}(\mu) &= X(\mu) \mid_{\Sigma\varpi_i}, \\ \mathfrak{Y}_i^{(0)}(\mu) &= Y(\mu) \mid_{\Sigma\varpi_i}. \end{aligned} \tag{37}$$

The X - and Y -functions here are to be evaluated for the albedo $\Sigma\varpi_i$, the sum of all the albedos. We have denoted this sum by means of an index in equations (37).

The first iterates are obtained by substituting the zeroth iterates in equations (23) and (24); we find

$$\mathfrak{X}_i^{(1)}(\mu) = 1 + \frac{\mu}{2} \sum_l \varpi_l \frac{k_l}{k_i} \int_0^1 \frac{d\mu'}{k_l \mu / k_i + \mu'} [X(\mu)X(\mu') - Y(\mu)Y(\mu')]_{\Sigma\varpi_l} \tag{38}$$

and

$$\begin{aligned} \mathfrak{Y}_i^{(1)}(\mu) &= e^{-k_i \tau / \mu} + \frac{\mu}{2} \sum_l \varpi_l \frac{k_l}{k_i} \int_0^1 \frac{d\mu'}{k_l \mu / k_i - \mu'} \\ &\quad [Y(\mu)X(\mu') - X(\mu)Y(\mu')]_{\Sigma\varpi_l}. \end{aligned} \tag{39}$$

With the notations

$$\xi^\pm(\mu) = \frac{\varpi\mu}{2} \int_0^1 \frac{X(\mu')}{\mu \pm \mu'} d\mu' \tag{40}$$

and

$$\zeta^\pm(\mu) = \frac{\varpi\mu}{2} \int_0^1 \frac{Y(\mu')}{\mu \pm \mu'} d\mu', \tag{41}$$

equations (38) and (39) become

$$\mathfrak{X}_i^{(1)}(\mu) = 1 + \frac{1}{\Sigma\varpi_i} \sum_i \varpi_i \left[X(\mu) \xi^+ \left(\frac{k_i}{k_i} \mu \right) - Y(\mu) \zeta^+ \left(\frac{k_i}{k_i} \mu \right) \right]_{\Sigma\varpi_i} \tag{42}$$

and

$$\mathfrak{Y}_i^{(1)}(\mu) = e^{-k_i\tau/\mu} + \frac{1}{\Sigma\varpi_i} \sum_i \varpi_i \left[Y(\mu) \xi^- \left(\frac{k_i}{k_i} \mu \right) - X(\mu) \zeta^- \left(\frac{k_i}{k_i} \mu \right) \right]_{\Sigma\varpi_i}, \tag{43}$$

from which the first iterates may be evaluated, when ξ^\pm and ζ^\pm are tabulated.

We have not been able to derive analytical expressions for the new functions ξ^\pm and ζ^\pm . A numerical integration may be needed. We have, however, established equation (46), below, which, together with the defining equations (35) and (36) for the X - and Y -functions, constitutes three relations between the four unknowns ξ^\pm and ζ^\pm . These three equations are

$$X(\mu) = 1 + X(\mu) \xi^+(\mu) - Y(\mu) \zeta^+(\mu), \tag{44}$$

$$Y(\mu) = e^{-k\tau/\mu} + Y(\mu) \xi^-(\mu) - X(\mu) \zeta^-(\mu), \tag{45}$$

and

$$\xi^+(\mu) + \xi^-(\mu) - \xi^+(\mu) \xi^-(\mu) + \zeta^+(\mu) \zeta^-(\mu) = \frac{\varpi\mu}{2} \ln \left| \frac{1 + \mu}{1 - \mu} \right|. \tag{46}$$

The derivation of equation (46) is given in the appendix.

IV. SOLUTION OF THE COUPLED TRANSFER EQUATION: SEMI-INFINITE ATMOSPHERES

a) The Integral Equation

Either by a direct method similar to that used in the case of finite atmospheres or by passing to the limit of large optical thicknesses, one could verify that

$$\mathfrak{X}_i(\mu) \rightarrow \mathfrak{S}_i(\mu) \quad \text{and} \quad \mathfrak{Y}_i(\mu) \rightarrow 0 \quad \text{as} \quad \tau \rightarrow \infty, \tag{47}$$

where $\mathfrak{S}_i(\mu)$ satisfies the integral equation,

$$\mathfrak{S}_i(\mu) = 1 + \frac{\mu}{2} \mathfrak{S}_i(\mu) \sum_i \varpi_i k_i \int_0^1 \frac{\mathfrak{S}_i(\mu')}{k_i\mu + k_i\mu'} d\mu'. \tag{48}$$

It is important to note that here the condition $\tau \rightarrow \infty$ physically means that the actual optical thicknesses, $k_i\tau$, for *all* frequencies must be large.

The $\mathfrak{S}_i(\mu)$ defined by equation (48) is again the expression for fluorescent scattering that corresponds to Chandrasekhar's H -function for ordinary scattering (*R.T.*, pp. 96 ff.). The latter satisfies the integral equation

$$H_i(\mu) = 1 + \frac{\mu}{2} \varpi_i H_i(\mu) \int_0^1 \frac{H_i(\mu')}{\mu + \mu'} d\mu'. \tag{49}$$

As τ tends to infinity, equations (15) and (16) reduce to

$$S_{ij}(\mu, \mu_0) = \frac{\varpi_i k_j \mu \mu_0}{k_j \mu + k_i \mu_0} \mathfrak{S}_i(\mu) \mathfrak{S}_j(\mu_0) \tag{50}$$

and

$$T_{ij}(\mu, \mu_0) = 0. \quad (51)$$

Scattering functions of the form $S_{ij}(\mu, \mu_0)$, depending on two frequencies, were first introduced by Busbridge and Stibbs (1954). Those authors, dealing with coupled multiplet transitions for line formation in semi-infinite atmosphere, have arrived at a scattering function equivalent to that in equation (50), involving functions essentially the same as our $\mathfrak{S}_i(\mu)$.

b) Iterative Solution of the \mathfrak{S} -Function

As in developing the \mathfrak{X} - and \mathfrak{Y} -functions, we start with the zeroth iterates,

$$\mathfrak{S}_1^{(0)}(\mu) = \mathfrak{S}_2^{(0)}(\mu) = \dots = H(\mu, \Sigma\varpi_l). \quad (52)$$

The first iterate is obtained by substituting the zeroth iterate in equation (48):

$$\mathfrak{S}_i^{(1)}(\mu) = 1 + \frac{\mu}{2} H(\mu, \Sigma\varpi_l) \sum_l \varpi_l \frac{k_l}{k_i} \int_0^1 \frac{H(\mu', \Sigma\varpi_l)}{k_l\mu/k_i + \mu'} d\mu'. \quad (53)$$

To eliminate the integral in this equation, we rewrite equation (49) as

$$\int_0^1 \frac{H(\mu') d\mu'}{\mu + \mu'} = \frac{2}{\varpi\mu} \left[1 - \frac{1}{H(\mu)} \right]. \quad (54)$$

Replacing μ by $k_l\mu/k_i$ and ϖ by $\Sigma\varpi_l$, we get

$$\int_0^1 \frac{H(\mu', \Sigma\varpi_l) d\mu'}{k_l\mu/k_i + \mu'} = \frac{2}{\Sigma\varpi_l} \frac{k_i}{k_l\mu} \left[1 - \frac{1}{H(k_l\mu/k_i, \Sigma\varpi_l)} \right]. \quad (55)$$

Substituting this expression in equation (53) gives

$$\mathfrak{S}_i^{(1)}(\mu) = 1 + H(\mu, \Sigma\varpi_l) \left[1 - \frac{1}{\Sigma\varpi_l} \sum_l \frac{\varpi_l}{H(k_l\mu/k_i, \Sigma\varpi_l)} \right], \quad (56)$$

which is in a convenient form for evaluation of the first iterates. The H -function in this equation, however, may have arguments greater than unity. If so, then equation (53) will be a more suitable expression to get $\mathfrak{S}_i^{(1)}(\mu)$ and a tabulation of

$$\int_0^1 \frac{H(\mu') d\mu'}{(\mu + \mu')}$$

for $\mu > 1$ will be needed.

c) A Numerical Example

The ground level of sodium, $3s \ ^2S_{1/2}$, consists of two hyperfine components, which split the D lines into the groups D_{1a} , D_{1b} , D_{2a} , and D_{2b} (see Chamberlain 1961, p. 429). (The hyperfine splitting in $3p \ ^2P$ levels is small and we neglect it [see Fig. 1]. The two sublevels have hyperfine quantum numbers $F = 2$ and 1.)

Assuming an optically thick atmosphere of sodium vapor, we consider the coupling between the hyperfine groups a and b (in either D_1 or D_2). Because of the small energy separation between the two sublevels of the ground level, the ratio of their populations at atmospheric temperatures will be that of their statistical weights, 5:3; but the two transitions D_a and D_b will have equal f -values and consequently equal absorption coefficients (per particle). Thus the ratio of their volume absorption coefficients will be $k_a:k_b = 5:3$. Einstein's coefficients for spontaneous emission for groups of a and b are also in the ratio 5:3, which, according to equation (5), gives $\varpi_a = \frac{5}{8}$ and $\varpi_b = \frac{3}{8}$.

From equations (9) and (50) we find

$$I_a(0, +\mu) = \frac{\varpi_a \mu_0}{4} \mathfrak{S}_a(\mu) \left[\frac{F_a}{\mu + \mu_0} \mathfrak{S}_a(\mu_0) + \frac{F_b}{\mu + k_a \mu_0 / k_b} \mathfrak{S}_b(\mu_0) \right] \quad (57)$$

and

$$I_b(0, +\mu) = \frac{\varpi_b \mu_0}{4} \mathfrak{S}_b(\mu) \left[\frac{F_a}{\mu + k_b \mu_0 / k_a} \mathfrak{S}_a(\mu_0) + \frac{F_b}{\mu + \mu_0} \mathfrak{S}_b(\mu_0) \right]. \quad (58)$$

The conditions on the ϖ 's and k 's are close to those of symmetric coupling. Equations (52) and (56) give

$$\mathfrak{S}_a^{(0)}(\mu) = \mathfrak{S}_b^{(0)}(\mu) = H(\mu, 1), \quad (59)$$

$$\mathfrak{S}_a^{(1)}(\mu) = H(\mu, 1) - \varpi_a \left[1 - \frac{H(\mu, 1)}{H(k_b \mu / k_a, 1)} \right], \quad (60)$$

and

$$\mathfrak{S}_b^{(1)}(\mu) = H(\mu, 1) - \varpi_b \left[1 - \frac{H(\mu, 1)}{H(k_a \mu / k_b, 1)} \right]. \quad (61)$$

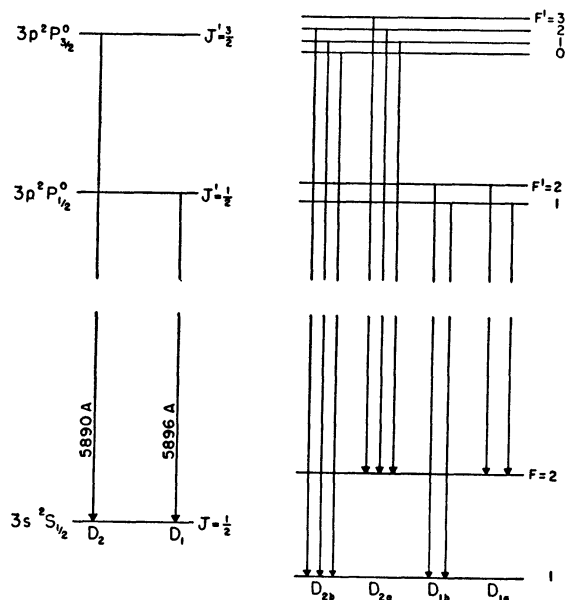


FIG. 1 —Energy levels of the 3s and 3p configurations of sodium, showing hyperfine structure. After Chamberlain (1961), p. 429.

In Figure 2, for $\mu_0 = 0.6$ and $F_a = F_b = 1$, we have plotted I_a and I_b (using both zeroth and first iterates of the \mathfrak{S} -function). Also, the non-coupled solution, which is simply the solution of equation (6) with $\varpi_a = \varpi_b = 1$ and which is given by

$$I_a(0, +\mu) = I_b(0, +\mu) = \frac{F \mu_0}{\mu + \mu_0} H(\mu, 1) H(\mu_0, 1), \quad (62)$$

is plotted in the figure. Presumably, $I_a^{(1)}$ and $I_b^{(1)}$ are in closer agreement with the true intensities than $I_a^{(0)}$ and $I_b^{(0)}$. Apparently, the use of the zeroth iterate of the \mathfrak{S} -function overestimates the effect of the coupling. As one would expect, more radiation is reflected in the a -component, which has a higher albedo, than in the b -component.

V. PRINCIPLE OF RECIPROCITY

From equations (15) and (50), one immediately sees that, in both finite and infinite atmospheres,

$$\frac{S_{ij}(\mu, \mu_0)}{\varpi_i k_j} = \frac{S_{ji}(\mu_0, \mu)}{\varpi_j k_i}, \quad (63)$$

which was established for the semi-infinite atmosphere by Busbridge and Stibbs. Apart from the factor $1/\varpi k$, a transposition of both pair of indices i, j , and variables, μ, μ_0 , leaves the scattering function unchanged. This property may be well interpreted in terms of the *principle of reciprocity* of Helmholtz, which was originally developed for geometrical optics. Chandrasekhar's extension of this principle to the case of ordinary scattering states that, upon an interchange of the directions of incidence and emergence (i.e., the interchange of μ and μ_0), the scattering function remains the same. (It should

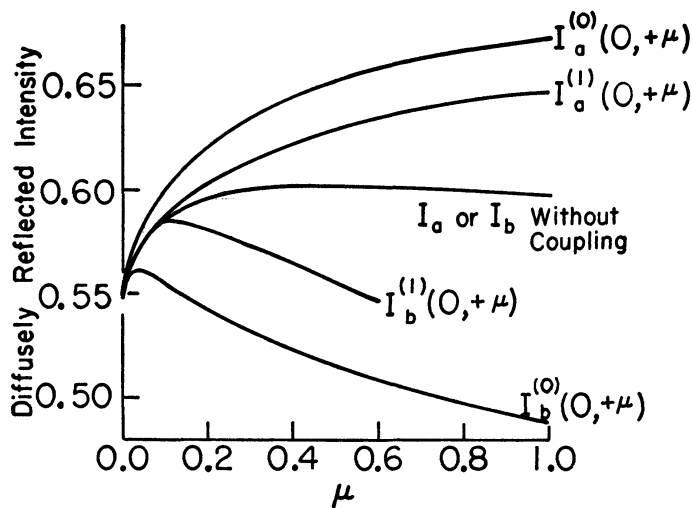


FIG. 2.—Diffusely reflected intensities (erg/cm² sec sterad unit-frequency-interval) of hyperfine components a and b of D_1 or D_2 lines of Na, from a semi-infinite atmosphere. The central curve gives I_a and I_b , with coupling between the two components disregarded. Curves are plotted for $\pi F_a = \pi F_b = \pi$ erg/cm² sec unit-frequency-interval, and $\mu_0 = 0.6$.

be noticed that his scattering function corresponds to our $S_{ij}[\mu, \mu_0]$ with all other elements identically zero.)

Let us recall from the definition of S_{ij} (eq. [9]) that

$$\mu I_{ij}(0, +\mu) = \frac{1}{4} F_j S_{ij}(\mu, \mu_0), \quad (64)$$

where $\mu I_{ij}(0, +\mu)$ is the component of the emergent flux I_{ij} normal to the plane of the atmosphere. This normal flux is in the frequency i and is due to fluorescent scattering of the incident flux πF_j in the frequency j . The principle of reciprocity, as applied to the fluorescent scattering, therefore states:

Upon replacing the incident flux F_j by F_i and interchanging the directions of incidence and emergence, apart from a constant factor $1/\varpi k$, the emergent normal flux remains the same. That is,

$$\frac{\mu I_{ij}(0, +\mu)}{\varpi_i k_j F_j} = \frac{\mu_0 I_{ji}(0, +\mu_0)}{\varpi_j k_i F_i}. \quad (65)$$

The origin of the factor $1/\varpi k$ may be traced back to single scattering. From the definition of k and ϖ we recall that k represents the absorption and ϖ shows the re-emis-

sion of the absorbed light. Whereas, for I_{ij} , absorption takes place in frequency j and re-emission in frequency i ; for I_{ji} , the inverse process is the case, which clarifies the origin of the proportionality constant in the reciprocity relation (63).

The principle of reciprocity stated above may be extended to incident and transmitted radiations, since, from equation (16), one has

$$\frac{T_{ij}(\mu, \mu_0)}{\varpi_i k_j} = \frac{T_{ji}(\mu_0, \mu)}{\varpi_j k_i}. \quad (66)$$

B. EFFECT OF CONTINUOUS ABSORPTION ON FLUORESCENT SCATTERING

I. COUPLED TRANSFER EQUATION WITH CONTINUOUS ABSORPTION

Let us recapitulate the problem of the fluorescent scattering. In Part A we assumed that an atmospheric constituent had several populated levels and scattered the radiation by successive absorptions and re-emissions. The line-absorption coefficient (per unit volume) was $\kappa_i^{(\nu)}$, and the albedo was ϖ_i , where i referred to the line or band in question and ν represented the dependence on the frequency within the profile.

Continuous absorption across a profile i can be accounted for by introducing a continuous volume absorption coefficient σ_i which is constant across the profile but may vary from one profile to another. The optical thickness, in analogy to equation (1), is

$$t_i = \int_z^\infty [\sigma_i(z) + \kappa_i(z)] dz, \quad (67)$$

where z is the altitude. (The subscript ν on t_i and κ_i has been suppressed.)

As a matter of convenience, however, as in Part A (cf. eqs. [2], [3], and [4]), we refer the optical thickness to the standard depth t , such that

$$dt_i = (s_i + k_i) dt = K_i dt, \quad (68)$$

where

$$k_i = \frac{\kappa_i}{\sum_i (\sigma_i + \kappa_i)}, \quad (69)$$

$$s_i = \frac{\sigma_i}{\sum_i (\sigma_i + \kappa_i)}, \quad (70)$$

and

$$dt = - \sum_i (\sigma_i + \kappa_i) dz. \quad (71)$$

For an atmosphere at a constant temperature and with the relative chemical abundances independent of height, all coefficients K_i , k_i , and s_i are independent of z . The variable t , independent of any particular value of i , is the sum of the optical thicknesses at corresponding positions in all lines participating in the absorption of the external radiation.

We assume that the radiation absorbed in the continuum is totally lost to the radiation field and is not re-emitted by inverse processes. In other words, we let the albedo of the continuous absorption be zero. With this consideration, the transfer equation (7) modifies to

$$\mu \frac{dI_i(t, \mu)}{dt} = K_i I_i(t, \mu) - \frac{\varpi_i}{2} \sum_j k_j \int_{-1}^1 I_j(t, \mu') d\mu' - \frac{\varpi_i}{4} \sum_j k_j F_j e^{-K_j t / \mu_0}. \quad (72)$$

On the right-hand side of this equation the first term—the absorbed radiation—is proportional to the total absorption coefficient K_i . But the re-emission terms (source function) are proportional to the k_j 's, the absorption coefficients in the lines only.

Corollary.—When continuous absorption is present, the analogue of equation (33) becomes

$$\frac{K_i}{\varpi_i} I_i(t, K_i \mu) = \frac{K_j}{\varpi_j} I_j(t, K_j \mu), \quad (73)$$

which can be verified by substitution in the transfer equation (72) (see the proof for eq. [24] of Paper I).

II. SOLUTION OF THE TRANSFER EQUATION (72)

The method of solution is the same as without continuous absorption. For the sake of brevity, we present the final results only.

a) Finite Atmosphere

The diffusely reflected and transmitted intensities, $I_i(0, +\mu)$ and $I_i(\tau, -\mu)$, are given by equations (9) and (10), where the scattering and transmission functions, $S_{ij}(\tau | \mu, \mu_0)$ and $T_{ij}(\tau | \mu, \mu_0)$, respectively, now satisfy these equations:

$$\left(\frac{K_i}{\mu} + \frac{K_j}{\mu_0} \right) S_{ij}(\tau | \mu, \mu_0) = \varpi_i k_j [\mathfrak{X}_i(\mu) \mathfrak{X}_j(\mu_0) - \mathfrak{Y}_i(\mu) \mathfrak{Y}_j(\mu_0)], \quad (74)$$

$$\left(\frac{K_i}{\mu} - \frac{K_j}{\mu_0} \right) T_{ij}(\tau | \mu, \mu_0) = \varpi_i k_j [\mathfrak{X}_i(\mu) \mathfrak{Y}_j(\mu_0) - \mathfrak{Y}_i(\mu) \mathfrak{X}_j(\mu_0)], \quad (75)$$

$$\frac{\partial S_{ij}(\tau | \mu, \mu_0)}{\partial \tau} = \varpi_i k_j \mathfrak{Y}_i(\mu) \mathfrak{Y}_j(\mu_0), \quad (76)$$

and

$$\left(\frac{K_j}{\mu_0} - \frac{K_i}{\mu} \right) \frac{\partial T_{ij}(\tau | \mu, \mu_0)}{\partial \tau} = \varpi_i k_j \left[\frac{K_j}{\mu_0} \mathfrak{X}_i(\mu) \mathfrak{Y}_j(\mu_0) - \frac{K_i}{\mu} \mathfrak{Y}_i(\mu) \mathfrak{X}_j(\mu_0) \right]. \quad (77)$$

The \mathfrak{X} - and \mathfrak{Y} -functions are defined by

$$\mathfrak{X}_i(\mu) = 1 + \frac{\mu}{2} \sum_l \varpi_l k_l \int_0^1 \frac{d\mu'}{K_l \mu + K_i \mu'} [\mathfrak{X}_i(\mu) \mathfrak{X}_l(\mu') - \mathfrak{Y}_i(\mu) \mathfrak{Y}_l(\mu')] \quad (78)$$

and

$$\mathfrak{Y}_i(\mu) = e^{-\kappa_i \tau / \mu} + \frac{\mu}{2} \sum_l \varpi_l k_l \int_0^1 \frac{d\mu'}{K_l \mu - K_i \mu'} [\mathfrak{Y}_i(\mu) \mathfrak{X}_l(\mu') - \mathfrak{X}_i(\mu) \mathfrak{Y}_l(\mu')]. \quad (79)$$

These \mathfrak{X} - and \mathfrak{Y} -functions are essentially the same as those of equations (23) and (25), provided that one replaces k_i with K_i and replaces ϖ_i with the reduced albedo, $k_i \varpi_i / K_i$, for all values of i . (This reduced albedo, which varies across the line profile, is the same albedo as in eq. [2] of Paper I, where it is denoted by $\varpi_{i, \nu}$). Likewise, the iterates of the above \mathfrak{X} - and \mathfrak{Y} -functions will be obtained from equations (37), (42), and (43) by making the changes in k_i and ϖ_i just stated.

b) Semi-infinite Atmosphere

As the optical thickness $K_i \tau$ (for all values of i) grows large, \mathfrak{Y}_i vanishes, and \mathfrak{X}_i tends to a limit of \mathfrak{S}_i , which satisfies the relation

$$\mathfrak{S}_i(\mu) = 1 + \frac{\mu}{2} \mathfrak{S}_i(\mu) \sum_l \varpi_l k_l \int_0^1 \frac{\mathfrak{S}_l(\mu')}{K_l \mu + K_i \mu'} d\mu'. \quad (80)$$

Again this \mathfrak{S} and its iterates are the same as those without continuous absorption (see eqs. [48], [52], and [56]), if one replaces the k_i 's by the K_i 's and the ϖ_i 's by the values $\varpi_i, \nu = k_i \varpi_i / K_i$.

The diffusely reflected intensity $I_i(0, +\mu)$ will be given by equation (9), and the scattering function will be

$$S_{ij}(\mu, \mu_0) = \frac{\varpi_i k_j \mu \mu_0}{K_j \mu + K_i \mu_0} \mathfrak{S}_i(\mu) \mathfrak{S}_j(\mu_0). \tag{81}$$

With these modifications, the theory follows that developed in Part A.

c) A Numerical Example

An example of line formation in the presence of continuous absorption is furnished by atmospheres composed of oxygen and nitrogen. At ordinary temperatures, nitrogen molecules are predominantly in the ground state, $X^1\Sigma_g^+$, and, upon absorbing and re-emitting ultraviolet sunlight (1450–1900 Å), may give rise to Lyman-Birge-Hopfield (LBH) bands (a $^1\Pi_g - X^1\Sigma_g^+$). At the same time this wavelength region is located in the O_2 Schumann-Runge continuum, where the absorbed radiation serves to dissociate

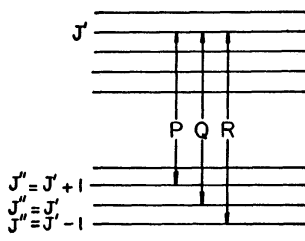


FIG. 3—A schematic diagram illustrating the coupling of P , Q , and R branches (with the same upper level) in a molecular (vibrational-rotational) transition.

molecular oxygen. We assume that the photons absorbed in the continuum are lost to the radiation field. In the following we investigate how the rotational structure of the LBH bands could be affected by such continuous absorption.

Being $\Pi-\Sigma$ transitions, the LBH bands have P , Q , and R branches. Any P , Q , R trio which have the same upper level J' will be coupled to one another (Fig. 3). The transitions have line strengths proportional to J' , $2J' + 1$, and $J' + 1$, respectively (see Herzberg 1950, p. 208). Thus, from equation (6), we find

$$\varpi_P = \frac{J'}{2(2J' + 1)} \varpi, \quad \varpi_Q = \frac{\varpi}{2}, \quad \text{and} \quad \varpi_R = \frac{J' + 1}{2(2J' + 1)} \varpi, \tag{82}$$

where $\varpi = \varpi_P + \varpi_Q + \varpi_R$ is the total albedo for the band in question.

The rotational levels are closely spaced, so that, at temperatures of about 300° K, the Boltzmann factor for any two of the lower levels of the P , Q , R trio is approximately unity. With this approximation, the volume absorption coefficient, κ , and, alternatively, the coefficient k becomes proportional to line strengths; that is,

$$k_P : k_Q : k_R = J' : (2J' + 1) : (J' + 1). \tag{83}$$

(For the appropriate formulae see Chamberlain 1961, pp. 17–18.)

With a constant coefficient for continuous absorption, we have, from equations (69) and (70), $s = r_P k_P$, where $r = \sigma / \kappa_P$. The total absorption coefficients become

$$\begin{aligned} K_P &= k_P + s = (1 + r_P) k_P, \\ K_Q &= k_Q + s = (1 + r_Q) k_Q, \end{aligned} \tag{84}$$

and

$$K_R = k_R + s = (1 + r_R) k_R.$$

Equations (83) furnish two relations among the ratios r_P , r_Q , and r_R .

We assume a semi-infinite atmosphere where the incident solar flux at all three frequencies is the same, i.e., $\pi F_P = \pi F_Q = \pi F_R = \pi F$. From equations (9), (73), and (81) we find

$$\begin{aligned} \frac{K_P}{\varpi_P} I_P(0, +K_P\mu) &= \frac{K_Q}{\varpi_Q} I_Q(0, +K_Q\mu) = \frac{K_R}{\varpi_R} I_R(0, +K_R\mu) \\ &= \frac{F\mu_0}{4} \mathfrak{S}_P(K_P\mu) \left[\frac{k_P \mathfrak{S}_P(\mu_0)}{K_P\mu + \mu_0} + \frac{k_Q \mathfrak{S}_Q(\mu_0)}{K_Q\mu + \mu_0} + \frac{k_R \mathfrak{S}_R(\mu_0)}{K_R\mu + \mu_0} \right]. \end{aligned} \quad (85)$$

The first iterates of the \mathfrak{S} -functions in these equations are given by equation (56), subject to replacing k by K and ϖ by $k\varpi/K$. Equations (85) provide the necessary relations to calculate the absolute or relative intensities of any P , Q , R trio in a band.

For $J' = 3$ of the 0-0 band of the LBH system and for the centers of the P , Q , and R profiles, the intensities of equation (85) are plotted versus μ in Figure 4 (*solid curves*). For the purpose of comparison, we have also plotted the same intensities in the absence of continuous absorption (*broken lines*). The albedo of the 0-0 band, $\varpi = 0.07$, is calculated from the Franck-Condon factors of the LBH system given by Jarman *et al.* (1955). The arbitrarily chosen constants are $\mu_0 = 0.5$, $\pi F = \pi$, and $r_P = 0.4$ ($r_Q = 6/41$ and $r_R = 0.3$ then follow from eqs. [83] and [84]). The H -functions to be used in equation (56) are calculated from the formulae and tables given by Stibbs and Weir (1959).

The ratio $r_P = 0.4$ used in this hypothetical example represents a very small relative abundance of oxygen to nitrogen—roughly one-thousandth of the relative abundance prevailing in the 100-km region of the earth's atmosphere. As can be seen from Figure 4, even this trace of oxygen has markedly reduced the reflected intensities. Indeed, it would not be surprising if, in the terrestrial atmosphere, continuous oxygen absorption should prevent the formation of the ultraviolet emission spectrum of the LBH bands.

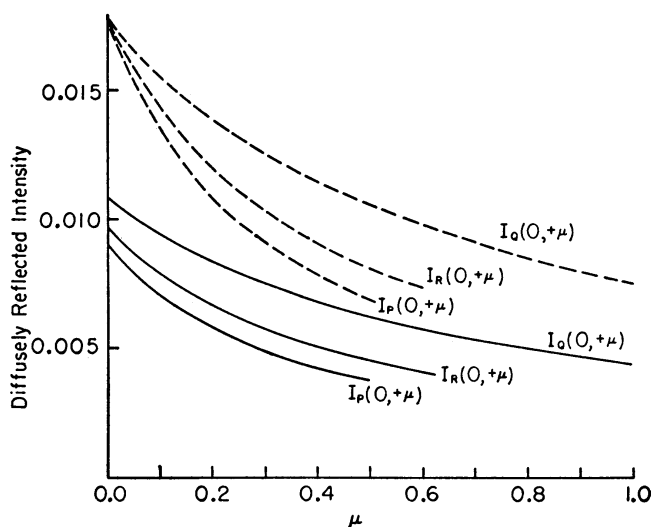


FIG. 4.—Diffusely reflected intensities (erg/cm² sec sterad unit-frequency-interval) of a P , Q , R trio of transitions with $J' = 3$ in the 0-0 band of the LBH system of N_2 . The atmosphere is taken to be semi-infinite, and the intensities are for the center of the line profiles. The fluxes are $\pi F_P = \pi F_Q = \pi F_R = \pi$ erg/cm² sec unit-frequency-interval, the albedo is $\varpi = \varpi_P + \varpi_Q + \varpi_R = 0.07$, and the incidence angle is $\mu_0 = 0.5$. Broken curves are for $r_P = r_Q = r_R = 0$, and the solid curves are for $r_P = 0.4$, $r_Q = 6/41$, and $r_R = 0.3$.

d) Principle of Reciprocity

As seen from equations (74), (75), and (81), the reciprocity relations of equations (63) and (66) also hold for fluorescent scattering with continuous absorption, and the statements and explanations of Section A, V apply here as well.

I wish to express my sincere thanks to Dr. J. W. Chamberlain for suggesting this problem and for his continuous interest and guidance in the course of research. Dr. M. H. Wrubel kindly read the manuscript and offered several suggestions toward its improvement.

APPENDIX

THE ξ - AND ζ -FUNCTIONS

In evaluating the first iterates of the \mathfrak{X} - and \mathfrak{Y} -functions, we introduced four functions $\xi^\pm(\mu)$ and $\zeta^\pm(\mu)$ (eqs. [40] and [41]). Also in Paper I we encountered the same functions in generalizing the definitions of the X - and Y -functions for arguments greater than unity. The latter functions were needed to derive the intensities of the fluorescent bands when continuous absorption is present. In the following we establish a relation between the four functions $\xi^\pm(\mu)$ and $\zeta^\pm(\mu)$.

Let us construct $\xi^+(\mu)$ directly by integrating the defining equation for $X(\mu)$, equation (35):

$$\begin{aligned} \xi^+(\mu) &= \frac{\varpi\mu}{2} \int_0^1 \frac{d\mu'}{\mu + \mu'} X(\mu') \\ &= \frac{\varpi\mu}{2} \int_0^1 \frac{d\mu'}{\mu + \mu'} \left\{ 1 + \frac{\varpi}{2} \mu' \int_0^1 \frac{d\mu''}{\mu' + \mu''} [X(\mu')X(\mu'') - Y(\mu')Y(\mu'')] \right\} \quad (\text{A1}) \\ &= \frac{\varpi\mu}{2} \ln \left| \frac{1 + \mu}{\mu} \right| + \frac{\varpi^2}{4} \mu \int_0^1 \int_0^1 \frac{\mu' d\mu' d\mu''}{(\mu + \mu')(\mu' + \mu'')} [X(\mu')X(\mu'') - Y(\mu')Y(\mu'')]. \end{aligned}$$

With the relation

$$\frac{\mu'}{(\mu + \mu')(\mu' + \mu'')} = \frac{\mu}{(\mu + \mu')(\mu - \mu'')} - \frac{\mu''}{(\mu - \mu'')(\mu' + \mu'')}, \quad (\text{A2})$$

equation (A1) becomes

$$\begin{aligned} \xi^+(\mu) &= \frac{\varpi\mu}{2} \ln \frac{1 + \mu}{\mu} + \xi^+(\mu) \xi^-(\mu) - \zeta^+(\mu) \zeta^-(\mu) \\ &\quad - \frac{\varpi^2}{4} \mu \int_0^1 \int_0^1 \frac{\mu'' d\mu' d\mu''}{(\mu - \mu'')(\mu' + \mu'')} [X(\mu')X(\mu'') - Y(\mu')Y(\mu'')]. \quad (\text{A3}) \end{aligned}$$

A similar direct integration for $\xi^-(\mu)$ gives

$$\begin{aligned} \xi^-(\mu) &= \frac{\varpi\mu}{2} \int_0^1 \frac{d\mu''}{\mu - \mu''} X(\mu'') \\ &= \frac{\varpi\mu}{2} \int_0^1 \frac{d\mu''}{\mu - \mu''} \left\{ 1 + \frac{\varpi}{2} \mu'' \int_0^1 \frac{d\mu'}{\mu' + \mu''} [X(\mu')X(\mu'') - Y(\mu')Y(\mu'')] \right\} \quad (\text{A4}) \\ &= -\frac{\varpi\mu}{2} \ln \left| \frac{1 - \mu}{\mu} \right| + \frac{\varpi^2}{4} \mu \int_0^1 \int_0^1 \frac{\mu'' d\mu' d\mu''}{(\mu - \mu'')(\mu' + \mu'')} \\ &\quad \times [X(\mu')X(\mu'') - Y(\mu')Y(\mu'')]. \end{aligned}$$

Adding equations (A3) and (A4) and rearranging the terms give the desired relation,

$$\xi^+(\mu) + \xi^-(\mu) - \xi^+(\mu)\xi^-(\mu) + \zeta^+(\mu)\zeta^-(\mu) = \frac{\omega\mu}{2} \ln \left| \frac{1+\mu}{1-\mu} \right|, \quad (\text{A5})$$

which is quoted as equation (46).

Note added in proof: A detailed discussion of principle of reciprocity for a more general case than the one discussed in this paper is given by K. M. Case (*Rev. Mod. Phys.*, **29**, 651, 1957).

REFERENCES

- Busbridge, I. W. 1955, *M N*, **115**, 661.
 Busbridge, I. W., and Stibbs, D. W. N. 1954, *M.N.*, **114**, 2.
 Chamberlain, J. W. 1961, *Physics of the Aurora and Airglow* (New York: Academic Press).
 Chamberlain, J. W., and Sobouti, Y. 1962, *Ap. J.*, **135**, 925 (Paper I).
 Chandrasekhar, S. 1950, *Radiative Transfer* (Oxford: Clarendon Press).
 Herzberg, G. 1950, *Spectra of Diatomic Molecules* (2d ed ; New York: D. Van Nostrand Co).
 Jarman, W. R., Fraser, P. A., and Nicholls, R. W. 1955, *Ap. J.*, **122**, 55.
 Stibbs, D. W. N., and Weir, R. E. 1959, *M.N.*, **119**, 512.