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SCATTERING AND TRANSMISSION FUNCTIONS FOR NON-COHERENT SCATTERING

YOUSEF SOBOUTI
Department of Physics, Pahlavi University, Shiraz, Iran
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ABSTRACT

Chandrasekhar's principles of invariance are used to solve the transfer equation of non-coherent scattering for the intensity of the diffusely reflected and transmitted radiation.

I. INTRODUCTION

Besides a computational approach to the problem of the non-coherent scattering of light, one finds two other techniques in the literature: (1) a method of principles of invariance, or a generalization of it, a method of invariant imbedding, and (2) a probabilistic technique originally formulated by Sobolev. Examples of the first technique are Busbridge (1953, 1955), Bellman, Kalaba, and Ueno (1962), and Sobouti (1962); examples of the second are Sobolev (1954), Ueno (1956, 1958*a*, *b*, *c*, *d*), and Ivanov (1963, 1965).

All these authors have invariably assumed isotropic and completely non-coherent scattering. They have found that the solutions are expressible in terms of an "*H*-function" in the case of semi-infinite atmospheres and in terms of a pair of "*X*- and *Y*-functions" in the case of finite atmospheres. In the following we relax the conditions of both isotropic scattering and complete non-coherence and solve the non-coherent transfer equation. The development requires an extension of Chandrasekhar's principles of invariance.

II. EQUATION OF TRANSFER

Let us consider a plane parallel atmosphere scattering in a non-coherent manner and illuminated by unidirectional radiation. The specific intensity of the diffuse radiation is governed by the following equation:

$$\mu \frac{dI_\nu(t,\boldsymbol{n})}{dt} = k_\nu I_\nu(t,\boldsymbol{n}) - \frac{1}{4\pi}\int_{\nu'}\int_{\Omega'} k_{\nu'} I_{\nu'}(t,\boldsymbol{n}') p_{\nu\nu'}(\boldsymbol{n},\boldsymbol{n}') d\nu' d\Omega' - \tfrac{1}{4}\int_{\nu_0} k_{\nu_0} F_{\nu_0} [\exp(-k_{\nu_0} t/\mu_0)] p_{\nu\nu_0}(\boldsymbol{n},-\boldsymbol{n}_0) d\nu_0 \,, \tag{1}$$

where the notation is as follows:

$\boldsymbol{n}$ = unit vector specifying a direction of radiation,

$\mu = \cos\theta$, where θ is the angle between $\boldsymbol{n}$ and the z-axis chosen perpendicular to the plane of the atmosphere,

$dt = -dz\int \kappa_\nu d\nu = \kappa dz$, where κ_ν is the volume absorption coefficient of the medium,

$k_\nu = \kappa_\nu/\kappa$, a normalized absorption coefficient ($k_\nu dt$ is then an element of the optical depth at frequency ν),

$I_\nu(t,\boldsymbol{n})$ = specific intensity of the diffuse radiation at the frequency ν, at the optical depth t, and in the direction $\boldsymbol{n}$,

πF_{ν_0} = radiation flux falling on the atmosphere in direction $-\boldsymbol{n}_0$ (a plus or minus sign preceding a direction vector indicates whether the direction in question is upward or downward with respect to the z-axis),

$p_{\nu\nu'}(\boldsymbol{n},\boldsymbol{n}')$ = phase function for single scattering. It is the fraction of energy absorbed from a beam of frequency ν', direction $\boldsymbol{n}'$ that scatters into frequency ν, direction $\boldsymbol{n}$,

$d\Omega'$ = element of solid angle around a direction vector $\boldsymbol{n}'$. The integration is to be effected over all solid angles Ω'.

The derivation of equation (1) may be found in the literature cited in § I. In §§ III and IV we shall discuss the phase function $p_{\nu\nu'}(\boldsymbol{n},\boldsymbol{n}')$ in some detail. Here we note only that this general form may cover cases of partial or complete redistribution in frequency. In particular, a partial non-coherence of the form $p_{\nu\nu'} = a_{\nu'}\delta(\nu - \nu') + b_{\nu\nu'}$ reduces equation (1) to that of Ambartsumyan (1958). Hummer (1962) critically examined the function $p_{\nu\nu'}$ (referred to as redistribution function) for moving particles and worked out a number of explicit examples.

III. A SOLUTION OF THE EQUATION OF NON-COHERENT SCATTERING

To obtain solutions for the diffuse radiation emerging from the atmosphere, one applies Chandrasekhar's principles of invariance with the necessary generalizations (cf. Chandrasekhar 1950). Let the diffusely reflected and transmitted radiation from the top and bottom of the atmosphere be given by

$$I_\nu(0,+\boldsymbol{n}) = \frac{1}{4\mu}\int F_{\nu_0} S_{\nu\nu_0}(\tau\,|\,\boldsymbol{n},\boldsymbol{n}_0)d\nu_0 \tag{2}$$

and

$$I_\nu(\tau,-\boldsymbol{n}) = \frac{1}{4\mu}\int F_{\nu_0} T_{\nu\nu_0}(\tau\,|\,\boldsymbol{n},\boldsymbol{n}_0)d\nu_0\,, \tag{3}$$

respectively, where τ is the total optical thickness of the atmosphere.

To obtain the scattering function $S_{\nu\nu_0}$, the transmission function $T_{\nu\nu_0}$, and their derivatives, we shall make use of the following four principles:

1. The intensity $I_\nu\,(t, +\boldsymbol{n})$ in the outward direction results from the diffuse reflection of the reduced incident flux, $\pi F_{\nu_0}\exp\,(-k_{\nu_0}t/\mu_0)$, and the diffuse reflection of the radiation $I_{\nu'}\,(t, -\boldsymbol{n}')$ incident on the atmosphere of optical thickness $(\tau - t)$ below t, in all frequencies ν'. That is:

$$\begin{aligned} I_\nu(t,+\boldsymbol{n}) = {} & \frac{1}{4\mu}\int_{\nu_0} F_{\nu_0}[\exp\,(-\,k_{\nu_0}t/\mu_0)]S_{\nu\nu_0}(\tau - t\,|\,\boldsymbol{n},\boldsymbol{n}_0)d\nu_0 \\ & + \frac{1}{4\pi\mu}\int_{\nu'}\int_{\Omega'} S_{\nu\nu'}(\tau - t\,|\,\boldsymbol{n},\boldsymbol{n}')I_{\nu'}(t,-\boldsymbol{n}')d\nu' d\Omega'\,. \end{aligned} \tag{4}$$

2. The intensity $I_\nu\,(t, -\boldsymbol{n})$ in the inward direction results from the transmission of the incident flux, πF_{ν_0}, by the atmosphere of thickness t, above the surface t, and the reflection by this same surface of the diffuse radiation incident on it from below, in all frequencies ν'. That is:

$$I_\nu(t,-n) = \frac{1}{4\mu}\int_{\nu_0} F_{\nu_0} T_{\nu\nu_0}(t\,|\,n,n_0)d\nu_0 + \frac{1}{4\pi\mu}\int_{\nu'}\int_{\Omega'} S_{\nu\nu'}(t\,|\,n,n')I_{\nu'}(t,+n')d\nu' d\Omega' \ . \quad (5)$$

3. The intensity $I_\nu(0,+n)$ of equation (2), reflected by the entire atmosphere, is equivalent to the diffuse reflection of the incident flux, πF_{ν_0}, by the atmosphere of optical thickness t and the transmission by this same atmosphere of the diffuse radiation $I_{\nu'}(t,+n')$ incident on the surface t from below. That is:

$$\frac{1}{4\mu}\int_{\nu_0} F_{\nu_0} S_{\nu\nu_0}(\tau\,|\,n,n_0)d\nu_0 = [\exp\,(-\,k_\nu t\,|\,\mu)]I_\nu(t,+n) + \frac{1}{4\mu}\int_{\nu} F_{\nu_0} S_{\nu\nu_0}(t\,|\,n,n_0)d\nu_0 + \frac{1}{4\pi\mu}\int_{\nu'}\int_{\Omega'} T_{\nu\nu'}(t\,|\,n,n')I_{\nu'}(t,+n')d\nu' d\Omega' \ . \quad (6)$$

4. The intensity $I_\nu(\tau,-n)$ of equation (3), transmitted by the entire atmosphere, is equivalent to the transmission of the reduced incident flux, $\pi F_{\nu_0}\exp(-k_{\nu_0}t/\mu_0)$, and the diffuse radiation $I_{\nu'}(t,-n')$ incident on the surface t by the atmosphere of thickness $(\tau - t)$ below t. That is:

$$\frac{1}{4\mu}\int_{\nu_0} F_{\nu_0} T_{\nu\nu_0}(\tau\,|\,n,n_0)d\nu_0 = [\exp\,(-\,k_\nu(\tau - t)/\mu)]I_\nu(t,-n) + \frac{1}{4\mu}\int_{\nu_0} F_{\nu_0}[\exp\,(-\,k_{\nu_0}t/\mu_0)]T_{\nu\nu_0}(\tau - t\,|\,n,n_0)d\nu_0 + \frac{1}{4\pi\mu}\int_{\nu'}\int_{\Omega'} T_{\nu\nu'}(\tau - t\,|\,n,n')I_{\nu'}(t,-n')d\nu' d\Omega' \ . \quad (7)$$

It should be noted that in the above equation and the subsequent ones the angle integration, $d\Omega'$, is carried out over the positive or the negative hemisphere, depending on whether the integrand in question contains outwardly or inwardly directed radiation, respectively.

To solve equations (4)–(7) for $S_{\nu\nu_0}$ and $T_{\nu\nu_0}$, one takes the following steps:

a) Differentiation with respect to t, followed by passing to the limit of $t = 0$ in equations (4) and (6), and to the limit of $t = \tau$ in equations (5) and (7).

b) Substitution for the derivatives dI_ν/dt (evaluated at $t = 0$ or $t = \tau$) from equation (1).

c) Expressing $I_{\nu'}(0,+n')$ and $I_{\nu'}(\tau,-n')$ in terms of the scattering and transmission functions of equations (2) and (3).

d) Equating the coefficients of F_{ν_0} in the final results.

For example, equation (4) can be reduced in the following manner. Differentiating with respect to t and letting $t = 0$ yield

$$\mu\left[\frac{d}{dt}I_\nu(t,+n)\right]_{t=0} = -\frac{1}{4\mu_0}\int_{\nu_0} F_{\nu_0}k_{\nu_0}S_{\nu\nu_0}(\tau\,|\,n,n_0)d\nu_0 - \tfrac{1}{4}\int_{\nu_0} F_{\nu_0}\frac{\partial}{\partial\tau}S_{\nu\nu_0}(\tau\,|\,n,n_0)d\nu_0 + \frac{1}{4\pi}\int_{\nu'}\int_{\Omega'} S_{\nu\nu'}(\tau\,|\,n,n') \times\left[\frac{d}{dt}I_{\nu'}(t,-n')\right]_{t=0} d\nu' d\Omega' \ . \quad (8)$$

The boundary condition $I_\nu(0,-n) = 0$ is used to derive equation (8). Another boundary condition is $I_\nu(\tau,+n) = 0$ and will be used to derive equations (12)–(14) below. From transfer equation (1), one gets

$$\begin{aligned}\left[\frac{d}{dt} I_{\nu'}(t,-n')\right]_{t=0} &= \frac{1}{4\mu'} \int_{\nu_0} F_{\nu_0} k_{\nu_0} p_{\nu'\nu_0}(-n',-n_0) d\nu_0 \\ &+ \frac{1}{4\pi\mu'} \int_{\nu''}\int_{\Omega''} k_{\nu''} p_{\nu'\nu''}(-n',+n'') I_{\nu''}(0,+n'') d\nu'' d\Omega'' \\ &= \frac{1}{4\mu'} \int_{\nu_0} F_{\nu_0} k_{\nu_0} p_{\nu'\nu_0}(-n',-n_0) d\nu_0 \\ &+ \frac{1}{16\pi\mu'} \int_{\nu_0}\int_{\nu''}\int_{\Omega''} F_{\nu_0} \frac{k_{\nu''}}{\mu''} p_{\nu'\nu''}(-n',+n'') \\ &\times S_{\nu''\nu_0}(n'',n_0) d\nu_0 d\nu'' d\Omega'' ,\end{aligned} \tag{9}$$

where the second equality is obtained by expressing $I_{\nu''}(0,+n'')$ in terms of the scattering function of equation (2). For brevity we have omitted τ in $S_{\nu\nu_0}(\tau|n,n_0)$ and will continue to do so in subsequent equations. By the same procedure it follows that

$$\begin{aligned}\mu\left[\frac{d}{dt} I_\nu(t,+n)\right]_{t=0} &= \tfrac{1}{4}\frac{k_\nu}{\mu} \int_{\nu_0} F_{\nu_0} S_{\nu\nu_0}(n,n_0) d\nu_0 \\ &- \tfrac{1}{4}\int_{\nu_0} F_{\nu_0} k_{\nu_0} p_{\nu\nu_0}(+n,-n_0) d\nu_0 \\ &- \frac{1}{16\pi} \int_{\nu_0}\int_{\nu''}\int_{\Omega''} F_{\nu_0} \frac{k_{\nu''}}{\mu''} p_{\nu\nu''}(+n,+n'') \\ &\times S_{\nu''\nu_0}(n'',n_0) d\nu_0 d\nu' d\Omega'' .\end{aligned} \tag{10}$$

On substituting expressions (9) and (10) in equation (8) and on equating the coefficients F_{ν_0}, one obtains

$$\begin{aligned}\left[\frac{k_\nu}{\mu} + \frac{k_{\nu_0}}{\mu}\right] S_{\nu\nu_0}(n,n_0) + \frac{\partial}{\partial\tau} S_{\nu\nu_0}(n,n_0) &= k_{\nu_0} p_{\nu\nu_0}(+n,-n_0) \\ &+ \frac{1}{4\pi} \int_{\nu''}\int_{\Omega''} \frac{k_{\nu''}}{\mu''} p_{\nu\nu''}(+n,+n'') S_{\nu''\nu_0}(n'',n_0) d\nu'' d\Omega'' \\ &+ \frac{1}{4\pi} \int_{\nu'}\int_{\Omega'} \frac{k_{\nu_0}}{\mu'} S_{\nu\nu'}(n,n') p_{\nu'\nu_0}(-n',-n_0) d\nu' d\Omega' \\ &+ \frac{1}{16\pi^2} \int_{\nu'}\int_{\nu''}\int_{\Omega'}\int_{\Omega''} \frac{k_{\nu''}}{\mu'\mu''} S_{\nu\nu'}(n,n') p_{\nu'\nu''}(-n',+n'') \\ &\times S_{\nu''\nu}(n'',n_0) d\nu' d\nu'' d\Omega' d\Omega'' .\end{aligned} \tag{11}$$

Similarly, equations (5)–(7) yield

$$\begin{aligned}\frac{\partial S_{\nu\nu_0}(n,n_0)}{\partial\tau} = &\left\{\exp\left[-\left(\frac{k_\nu}{\mu}+\frac{k_{\nu_0}}{\mu_0}\right)\tau\right]\right\}k_{\nu_0}p_{\nu\nu_0}(+n,-n_0)\\ &+\frac{1}{4\pi}e^{-k_\nu\tau/\mu}\int_{\nu''}\int_{\Omega''}\frac{k_{\nu''}}{\mu''}p_{\nu\nu''}(+n,-n'')T_{\nu''\nu_0}(n'',n_0)d\nu''d\Omega''\\ &+\frac{1}{4\pi}e^{-k_{\nu_0}\tau/\mu_0}\int_{\nu'}\int_{\Omega'}\frac{k_{\nu_0}}{\mu'}T_{\nu\nu'}(n,n')p_{\nu'\nu_0}(+n',-n_0)d\nu'd\Omega'\\ &+\frac{1}{16\pi^2}\int_{\nu'}\int_{\nu''}\int_{\Omega'}\int_{\Omega''}\frac{k_{\nu''}}{\mu'\mu''}T_{\nu\nu'}(n,n')p_{\nu'\nu''}(+n',-n'')\\ &\times T_{\nu''\nu_0}(n'',n_0)d\nu'd\nu''d\Omega'd\Omega'',\end{aligned} \tag{12}$$

$$\begin{aligned}\frac{k_\nu}{\mu}T_{\nu\nu_0}(n,n_0)+\frac{\partial T_{\nu\nu_0}(n,n_0)}{\partial\tau} = &[\exp(-k_{\nu_0}\tau/\mu_0)]k_{\nu_0}p_{\nu\nu_0}(-n,-n_0)\\ &+\frac{1}{4\pi}\int_{\nu''}\int_{\Omega''}\frac{k_{\nu''}}{\mu''}p_{\nu\nu''}(-n,-n'')T_{\nu''\nu_0}(n'',n_0)d\nu''d\Omega''\\ &+\frac{1}{4\pi}[\exp(-k_{\nu_0}\tau/\mu_0)]\int_{\nu'}\int_{\Omega'}\frac{k_{\nu_0}}{\mu'}S_{\nu\nu'}(n,n')p_{\nu'\nu_0}(+n',-n_0)d\nu'd\Omega'\\ &+\frac{1}{16\pi^2}\int_{\nu'}\int_{\nu''}\int_{\Omega'}\int_{\Omega''}\frac{k_{\nu''}}{\mu'\mu''}S_{\nu\nu'}(n,n')p_{\nu'\nu''}(+n',-n'')\\ &\times T_{\nu''\nu_0}(n'',n_0)d\nu'd\nu''d\Omega'd\Omega'',\end{aligned} \tag{13}$$

and

$$\begin{aligned}\frac{k_{\nu_0}}{\mu_0}T_{\nu\nu_0}(n,n_0)+\frac{\partial T_{\nu\nu_0}(n,n_0)}{\partial\tau} = &e^{-k_\nu\tau/\mu}k_{\nu_0}p_{\nu\nu_0}(-n,-n_0)\\ &+\frac{1}{4\pi}e^{-k_\nu\tau/\mu}\int_{\nu''}\int_{\Omega''}\frac{k_{\nu''}}{\mu''}p_{\nu\nu''}(-n,+n'')S_{\nu''\nu_0}(n'',n_0)d\nu''d\Omega''\\ &+\frac{1}{4\pi}\int_{\nu'}\int_{\Omega'}\frac{k_{\nu_0}}{\mu'}T_{\nu\nu'}(n,n')p_{\nu'\nu_0}(-n',-n_0)d\nu'd\Omega'\\ &+\frac{1}{16\pi^2}\int_{\nu'}\int_{\nu''}\int_{\Omega'}\int_{\Omega''}\frac{k_{\nu''}}{\mu'\mu''}T_{\nu\nu'}(n,n')p_{\nu'\nu''}(-n',+n'')\\ &\times S_{\nu''\nu_0}(n'',n_0)d\nu'd\nu''d\Omega'd\Omega''.\end{aligned} \tag{14}$$

For layers of infinite optical thickness $T_{\nu\nu_0}$, $\partial S_{\nu\nu_0}/\partial\tau$, and $\partial T_{\nu\nu_0}/\partial\tau$ tend to zero. Equation (11), the only one of equations (11)–(14) remaining, reduces to

$$
\begin{aligned}
\left(\frac{k_\nu}{\mu}+\frac{k_{\nu_0}}{\mu_0}\right)S_{\nu\nu_0}(n,n_0) = {} & k_{\nu_0}p_{\nu\nu_0}(+n,-n_0) \\
& + \frac{1}{4\pi}\int_{\nu''}\int_{\Omega''}\frac{k_{\nu''}}{\mu''}p_{\nu\nu''}(+n,+n'')S_{\nu''\nu_0}(n'',n_0)d\nu''d\Omega'' \\
& + \frac{1}{4\pi}\int_{\nu'}\int_{\Omega'}\frac{k_{\nu_0}}{\mu'}S_{\nu\nu'}(n,n')p_{\nu'\nu_0}(-n',-n_0)d\nu'd\Omega' \\
& + \frac{1}{16\pi^2}\int_{\nu'}\int_{\nu''}\int_{\Omega'}\int_{\Omega''}\frac{k_{\nu''}}{\mu'\mu''}S_{\nu\nu'}(n,n')p_{\nu'\nu''}(-n'+n'') \\
& \times S_{\nu''\nu_0}(n'',n_0)d\nu'd\nu''d\Omega'd\Omega'' .
\end{aligned} \tag{15}
$$

In actual physical problems, the phase function satisfies the following reciprocity equation

$$k_{\nu'}p_{\nu\nu'}(n,n') = k_\nu p_{\nu'\nu}(-n',-n) , \tag{16}$$

which states that the probability of absorption of a (ν',n') photon followed by the emission of a (ν,n) photon is equal to the probability of the inverse process. If one further requires that $p_{\nu\nu'}(n,n')$ is a function of the angle between the directions n and n' alone, irrespective of the sign of the angle, one finds that

$$p_{\nu\nu'}(n,n') = p_{\nu\nu'}(-n,-n') = p_{\nu\nu'}(n',n) . \tag{17}$$

Let us define the transposed of a function $\psi_{\nu\nu'}(n,n')$ as

$$\bar{\psi}_{\nu\nu'}(n,n') = \psi_{\nu'\nu}(n',n) . \tag{18}$$

Following Chandrasekhar (1950), we prove that the scattering and transmission functions are respectively equal to their transposed functions. From the definitions of diffuse reflection and transmission, it follows that $S_{\nu\nu_0}(n,n_0)$ and $T_{\nu\nu_0}(n,n_0)$ tend to zero as $\tau \to 0$. Hence, from equations (11) and (12) we get the following asymptotic formulas $(\tau \to 0)$:

$$\left(\frac{k_\nu}{\mu}+\frac{k_{\nu_0}}{\mu_0}\right)S_{\nu\nu_0}(n,n_0) = k_{\nu_0}p_{\nu\nu_0}(+n,-n_0)\left\{1 - \exp\left[-\left(\frac{k_\nu}{\mu}+\frac{k_{\nu_0}}{\mu_0}\right)\tau\right]\right\} \tag{19}$$

and

$$\frac{\partial}{\partial\tau}S_{\nu\nu_0}(n,n_0) = k_{\nu_0}p_{\nu\nu_0}(+n,-n_0)\exp\left[-\left(\frac{k_\nu}{\mu}+\frac{k_{\nu_0}}{\mu_0}\right)\tau\right] . \tag{20}$$

These equations show that, in view of the reciprocity equation (16), $S_{\nu\nu_0}(n,n_0)$ and its derivative are equal to their transposed functions for small values of τ. Hence, by a series expansion of $S_{\nu\nu_0}$ and $S'_{\nu\nu_0}$ about a particular small τ, one deduces that they are equal for $\tau + d\tau$ and therefore for all values of τ. With the same reasoning, one also proves the equality of $T_{\nu\nu_0}$ and $\bar{T}_{\nu\nu_0}$.

Equations (11)–(15) yield exact solutions for the problem of non-coherent scattering. As far as the author is aware, they have not appeared in the literature before. Their prohibitively complicated appearance may seem to limit their practical application. In practice, however, their non-linear character proves more apt for numerical calculations than the original linear equation (1). They are particularly suitable for treatment by iterative methods. For example, one may take the first terms on the right sides of equations (11)–(15) as the first iterates for the corresponding expressions on the left sides and then readily evaluate the second iterates. Since the first iterates thus chosen are actually the solutions of the single scattering case, the second iterates lead to reasonably accurate results when τ is small.

IV. A REDUCTION OF EQUATIONS (11)–(14)

Let γ denote the angle between the two directions $\boldsymbol{n}$ and $\boldsymbol{n}'$. In view of equation (17), one may write the following Legendre expansion:

$$\begin{aligned} k_{\nu'}p_{\nu\nu'}(\boldsymbol{n},\boldsymbol{n}') &= \sum_{n}^{N} a_{\nu\nu'}{}^{(n)}P_n(\cos\gamma) \\ &= \sum_{m=0}^{N}\sum_{n=m}^{N}(2-\delta_{0m})a_{\nu\nu'}{}^{(n)}(-1)^m\frac{(n-m)!}{(n+m)!} \\ &\quad\times P_n{}^m(\mu)P_n{}^m(\mu')\cos m(\phi-\phi') \\ &= \sum_{m=0}^{N} a_m p_{\nu\nu'}{}^{(m)}(\mu,\mu')\cos m(\phi-\phi')\,, \end{aligned} \tag{21}$$

where the second equality follows from the addition theorem for $P_n(\cos\gamma)$, the last equality defines the symbols $p_{\nu\nu'}{}^{(m)}(\mu,\mu')$ and $a_m = (2-\delta_{0m})$. From the reciprocity equation (16), one concludes that

$$a_{\nu\nu'}{}^{(n)} = a_{\nu'\nu}{}^{(n)} \tag{22a}$$

and

$$p_{\nu\nu'}{}^{(m)}(\mu,\mu') = p_{\nu\nu'}{}^{(m)}(-\mu,-\mu') = p_{\nu'\nu}{}^{(m)}(\mu,\mu')\,. \tag{22b}$$

In view of the expansion of equation (21), the scattering and transmission functions may correspondingly be expanded:

$$S_{\nu\nu'}(\boldsymbol{n},\boldsymbol{n}') = \sum_{m=0}^{N} S_{\nu\nu'}{}^{(m)}(\mu,\mu')\cos m(\phi-\phi') \tag{23}$$

and

$$T_{\nu\nu'}(\boldsymbol{n},\boldsymbol{n}') = \sum_{m=0}^{N} T_{\nu\nu'}{}^{(m)}(\mu,\mu')\cos m(\phi-\phi')\,. \tag{24}$$

Hereafter we shall suppress the superscript m and denote a_m by a. By substituting these expansions in equations (11)–(14) and equating the coefficients of $\cos m(\phi-\phi')$ on both sides, one derives the following equations for $S_{\nu\nu'}{}^{(m)}$, $T_{\nu\nu'}{}^{(m)}$ and their derivatives:

$$\begin{aligned} \left(\frac{k_\nu}{\mu}+\frac{k_{\nu_0}}{\mu_0}\right)S_{\nu\nu_0}(\mu,\mu_0) + \frac{\partial S_{\nu\nu_0}(\mu,\mu_0)}{\partial\tau} &= a p_{\nu\nu_0}(\mu,-\mu_0) \\ &+ \tfrac{1}{2}\int_{\nu''}\int_0^1 p_{\nu\nu''}(\mu,\mu'')S_{\nu''\nu_0}(\mu'',\mu_0)d\nu''\frac{d\mu''}{\mu''} \\ &+ \tfrac{1}{2}\int_{\nu'}\int_0^1 S_{\nu\nu'}(\mu,\mu')p_{\nu'\nu_0}(-\mu',-\mu_0)d\nu'\frac{d\mu'}{\mu'} \\ &+ \frac{1}{4a}\int_{\nu'}\int_{\nu''}\int_0^1\int_0^1 S_{\nu\nu'}(\mu,\mu')p_{\nu'\nu''}(-\mu',\mu'') \\ &\times S_{\nu''\nu_0}(\mu'',\mu_0)d\nu' d\nu''\frac{d\mu'}{\mu'}\frac{d\mu''}{\mu''}\,, \end{aligned} \tag{25}$$

$$\frac{\partial S_{\nu\nu_0}(\mu,\mu_0)}{\partial\tau} = \left\{\exp\left[-\left(\frac{k_\nu}{\mu}+\frac{k_{\nu_0}}{\mu_0}\right)\tau\right]\right\} a p_{\nu\nu_0}(\mu,-\mu_0)$$
$$+\tfrac{1}{2}e^{-k_\nu\tau/\mu}\int_{\nu''}\int_0^1 p_{\nu\nu''}(\mu,-\mu'')T_{\nu''\nu_0}(\mu'',\mu_0)d\nu''\frac{d\mu''}{\mu''}$$
$$+\tfrac{1}{2}[\exp(-k_{\nu_0}\tau/\mu_0)]\int_{\nu'}\int_0^1 T_{\nu\nu'}(\mu,\mu')p_{\nu'\nu_0}(\mu',-\mu_0)d\nu'\frac{d\mu'}{\mu'} \tag{26}$$
$$+\frac{1}{4a}\int_{\nu'}\int_{\nu''}\int_0^1\int_0^1 T_{\nu\nu'}(\mu,\mu')p_{\nu'\nu''}(\mu',-\mu'')T_{\nu''\nu_0}(\mu'',\mu_0)d\nu' d\nu''\frac{d\mu'}{\mu'}\frac{d\mu''}{\mu''},$$

$$\frac{k_\nu}{\mu}T_{\nu\nu_0}(\mu,\mu_0)+\frac{\partial T_{\nu\nu_0}(\mu,\mu_0)}{\partial\tau} = [\exp(-k_{\nu_0}\tau/\mu_0)]a p_{\nu\nu_0}(\mu,\mu_0)$$
$$+\tfrac{1}{2}\int_{\nu''}\int_0^1 p_{\nu\nu''}(\mu,\mu'')T_{\nu''\nu_0}(\mu'',\mu_0)d\nu''\frac{d\mu''}{\mu''}$$
$$+\tfrac{1}{2}[\exp(-k_{\nu_0}\tau/\mu_0)]\int_{\nu'}\int_0^1 S_{\nu\nu'}(\mu,\mu')p_{\nu'\nu_0}(\mu',-\mu_0)d\nu'\frac{d\mu'}{\mu'} \tag{27}$$
$$+\frac{1}{4a}\int_{\nu'}\int_{\nu''}\int_0^1\int_0^1 S_{\nu\nu'}(\mu,\mu')p_{\nu'\nu''}(\mu',-\mu'')T_{\nu''\nu_0}(\mu'',\mu_0)d\nu' d\nu''\frac{d\mu'}{\mu'}\frac{d\mu''}{\mu''},$$

and

$$\frac{k_{\nu_0}}{\mu_0}T_{\nu\nu_0}(\mu,\mu_0)+\frac{\partial T_{\nu\nu_0}(\mu,\mu_0)}{\partial\tau} = e^{-k_\nu\tau/\mu}a p_{\nu\nu_0}(\mu,\mu_0)$$
$$+\tfrac{1}{2}e^{-k_\nu\tau/\mu}\int_{\nu''}\int_0^1 p_{\nu\nu''}(-\mu,\mu'')S_{\nu''\nu_0}(\mu'',\mu_0)d\nu''\frac{d\mu''}{\mu''}$$
$$+\tfrac{1}{2}\int_{\nu'}\int_0^1 T_{\nu\nu'}(\mu,\mu')p_{\nu'\nu_0}(\mu',\mu_0)d\nu'\frac{d\mu'}{\mu'} \tag{28}$$
$$+\frac{1}{4a}\int_{\nu'}\int_{\nu''}\int_0^1\int_0^1 T_{\nu\nu'}(\mu,\mu')p_{\nu'\nu''}(-\mu',\mu'')S_{\nu''\nu_0}(\mu'',\mu_0)d\nu' d\nu''\frac{d\mu'}{\mu'}\frac{d\mu''}{\mu''}.$$

It should be noticed that the components $S_{\nu\nu'}(\mu,\mu')$ and $T_{\nu\nu'}(\mu,\mu')$, like the original scattering and transmission functions, remain invariant under transposition.

V. SPECIAL CASE OF SEPARABLE PHASE FUNCTIONS

The case of isotropic scattering with complete redistribution in frequency has a particularly simple phase function, such that the expression $k_{\nu'}p_{\nu\nu'}(\boldsymbol{n},\boldsymbol{n}')$ (actually independent of $\boldsymbol{n}$ and $\boldsymbol{n}'$) factors into two terms, each depending on either variable ν or ν' (see, e.g., Sobolev 1956). This same situation, i.e., separation of the pair of variables (μ,ν) and (μ',ν'), may exist for some of the components, $p_{\nu\nu'}(\mu,\mu')$, of the phase function in equations (25)–(28). For example, if $a_{\nu\nu'}{}^{(N)}$ of equation (21) factors, then so does the corresponding $p_{\nu\nu'}{}^{(N)}(\mu,\mu')$. A similar separation of μ and μ' actually does take place in some coherent scattering problems (see Chandrasekhar 1950).

Let us assume that some $p_{\nu\nu'}(\mu,\mu')$ takes the following form:

$$p_{\nu\nu'}(\mu,\mu') = \psi_\nu(\mu)\psi_{\nu'}(\mu') = \psi_\nu(-\mu)\psi_{\nu'}(-\mu'). \tag{29}$$

The second equality follows from equation (22b). Substitution of equation (29) in equations (25)–(28), using the symmetry properties of $S_{\nu\nu_0}(\mu,\mu_0)$ and $T_{\nu\nu_0}(\mu,\mu_0)$ under transposition, gives

$$\left(\frac{k_\nu}{\mu}+\frac{k_{\nu_0}}{\mu_0}\right)S_{\nu\nu_0}(\mu,\mu_0)+\frac{\partial S_{\nu\nu_0}(\mu,\mu_0)}{\partial\tau}=a\psi_\nu(\mu)\psi_{\nu_0}(-\mu_0)X_\nu(\mu)X_{\nu_0}(\mu_0)\,, \tag{30}$$

$$\frac{\partial S_{\nu\nu_0}(\mu,\mu_0)}{\partial\tau}=a\psi_\nu(\mu)\psi_{\nu_0}(-\mu_0)Y_\nu(\mu)Y_{\nu_0}(\mu_0)\,, \tag{31}$$

$$\frac{k_\nu}{\mu}T_{\nu\nu_0}(\mu,\mu_0)+\frac{\partial T_{\nu\nu_0}(\mu,\mu_0)}{\partial\tau}=a\psi_\nu(\mu)\psi_{\nu_0}(\mu_0)X_\nu(\mu)Y_{\nu_0}(\mu_0) \tag{32}$$

and

$$\frac{k_{\nu_0}}{\mu_0}T_{\nu\nu_0}(\mu,\mu_0)+\frac{\partial T_{\nu\nu_0}(\mu,\mu_0)}{\partial\tau}=a\psi_\nu(\mu)\psi_{\nu_0}(\mu_0)Y_\nu(\mu)X_{\nu_0}(\mu_0)\,, \tag{33}$$

where

$$X_\nu(\mu)=1+\frac{1}{2a}\int_{\nu'}\int_0^1 S_{\nu\nu'}(\mu,\mu')\frac{\psi_{\nu'}(-\mu')}{\psi_\nu(\mu)}\frac{d\mu'}{\mu'}\,d\nu' \tag{34}$$

and

$$Y_\nu(\mu)=e^{-k_\nu\tau/\mu}+\frac{1}{2a}\int_{\nu'}\int_0^1 T_{\nu\nu'}(\mu,\mu')\frac{\psi_{\nu'}(\mu')}{\psi_\nu(\mu)}\frac{d\mu'}{\mu'}\,d\nu'\,. \tag{35}$$

On eliminating the derivatives of S and T from equations (30)–(33), one obtains

$$\left(\frac{k_\nu}{\mu}+\frac{k_{\nu_0}}{\mu_0}\right)S_{\nu\nu_0}(\mu,\mu_0)=a\psi_\nu(\mu)\psi_{\nu_0}(-\mu_0)[X_\nu(\mu)X_{\nu_0}(\mu_0)-Y_\nu(\mu)Y_{\nu_0}(\mu_0)] \tag{36}$$

and

$$\left(\frac{k_\nu}{\mu}-\frac{k_{\nu_0}}{\mu_0}\right)T_{\nu\nu_0}(\mu,\mu_0)=a\psi_\nu(\mu)\psi_{\nu_0}(\mu_0)[X_\nu(\mu)Y_{\nu_0}(\mu_0)-Y_\nu(\mu)(X_{\nu_0}\mu_0)]\,. \tag{37}$$

If we use equations (36) and (37) to substitute for $S_{\nu\nu_0}$ and $T_{\nu\nu_0}$ in equations (34) and (35), we obtain

$$X_\nu(\mu)=1+\frac{\mu}{2}\int_{\nu'}\int_0^1\frac{\psi_{\nu'}(\mu')\psi_{\nu'}(-\mu')}{k_{\nu'}\mu+k_\nu\mu'}[X_\nu(\mu)X_{\nu'}(\mu')-Y_\nu(\mu)Y_{\nu'}(\mu')]d\nu'd\mu' \tag{38}$$

and

$$Y_\nu(\mu)=e^{-k_\nu\tau/\mu}+\frac{\mu}{2}\int_{\nu'}\int_0^1\frac{\psi^2{}_{\nu'}(\mu')}{k_{\nu'}\mu-k_\nu\mu'}[Y_\nu(\mu)X_{\nu'}(\mu')-X_\nu(\mu)Y_{\nu'}(\mu')]d\nu'd\mu'\,. \tag{39}$$

Sobouti (1962) considered the problem of isotropic, non-coherent scattering in discrete frequencies with or without absorption in the continuum, and he derived a set of functions that emerge as special cases of the X_ν and Y_ν of equations (38) and (39) above. Ivanov (1963) studied the problem of isotropic, non-coherent scattering without continuous absorption and derived another special case of the same equations. Ivanov used Sobolev's technique of deriving an equation for the probability of quantum escape from the scattering medium. This method, although quite elegant and concise in solving isotropic scattering problems, has not been demonstrated to possess the generality and flexibility of Chandrasekhar's method in handling problems with any phase function of the form $p_{\nu\nu'}(n,n')$ or in dealing with questions of polarization.

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