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Pure Perturbation Spectra of Convectively Neutral Fluids

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Summary. A perturbing force may remove the degeneracy of the neutral state of a convecting fluid, giving rise to a sequence of very long period oscillations. As an example it is shown that a force-free magnetic field is capable of generating pure hydromagnetic oscillations with periods of the order of Alfven crossing-times. Stability of the perturbation sequence is intimately related with convective stability of the entire and perturbed fluid.

Key words: convection — pulsation

I. Indroduction

As a subadiabatic fluid evolves into a superadiabatic one the stable nature of its g-oscillations changes to an unstable one. The transition is smooth and continuous such that in the limiting case of an adiabatic structure the entire g-spectrum is neutral. The fluid motions corresponding to neutral and unstable g-modes coincide with those commonly associated with convective displacements of the fluid. An elaboration of these statements with proper mathematical reasoning can be found in Sobouti (1976).

Consider an adiabatic fluid. In addition to the neutral g-modes, the fluid possesses neutral toroidal displacements. These motions arise from the shear-free nature of the fluid and are always present. The two sequences of convective and toroidal neutral modes form a degenerate state of the fluid. A perturbing force, in general, is capable of removing part or all of the degeneracy, giving rise to a new sequence of oscillations. The squares of the frequencies of the new spectrum are of the order of the ratio of the perturbation energy to the moment of inertia of the fluid. Therefore, compared with acoustic modes, the oscillations have very long periods.

Furthermore, the equation governing the perturbations of the neutral modes is, in its own right, an eigenvalue equation independent from the p-modes of the system. This enables one to employ an eigenvalue technique, in contrast with the perturbation analysis of the non-neutral p-modes that ordinarily requires a perturbation procedure.

In Sections II–V a mathematical analysis of the problem is attempted. The statements made above are proved and a computational procedure is developed. In Sections VI–IX a force-free magnetic field is considered as the perturbing force. It is shown that the combined convective and toroidal neutral modes evolve into a spectrum of pure hydromagnetic oscillations. The eigenfrequencies and the eigenvectors of axially symmetric modes are calculated by a Rayleigh-Ritz variational procedure. The corresponding periods are, of course, of the order of Alfvèn crossing-times. In Section X the relevance of the present analysis to convective stability of a perturbed configuration is discussed and an alternative formulation of the stability criterion is suggested.

Some properties of g- and p-modes were investigated in an earlier paper [Sobouti (1977), hereafter referred to as "paper I"]. The present analysis is a continuation of the latter in that it explores the behavior of the same modes when subjected to a perturbing force. Extensive use has been made of the formalism developed in the latter reference

II. The Unperturbed System

In paper I a general Lagrangian displacement of a self-gravitating fliud was expanded in terms of two independent sets of basis-vectors, ζ_g and ζ_p. Of these the former emphasized the g-character of the displacement and the latter reflected its p-nature. The equation governing the eigenfrequencies and the eigenvectors was expressed as a matrix equation

$$\boldsymbol{T}^0\boldsymbol{Z}^0=\boldsymbol{S}^0\boldsymbol{Z}^0\boldsymbol{E}^0\,, \tag{1}$$

where $\boldsymbol{T}^0$ and $\boldsymbol{S}^0$ were constructed from the differential equation governing the Lagrangian displacement, $\boldsymbol{Z}^0$

was the matrix of eigenvectors (i.e. the matrix of the expansion coefficients referred to above), and E^0 was the diagonal matrix of eigenvalues (square of the eigenfrequencies). The use of two distinct sets of basis vectors allowed a partitioning of the matrices into *gg*-, *gp*-, *pg*- and *pp*-blocks. Thus:

$$A=\left[\begin{array}{c|c}A_{gg} & A_{gp}\\ \hline A_{pg} & A_{pp}\end{array}\right],\qquad A=T^0, S^0, Z^0, E^0\,. \tag{2}$$

It was further shown that in an adiabatic fluid, the subject matter of this paper, the *g*- and *p*-spectra became independent from one another. All *g*-frequencies vanished and the corresponding *g*-eigenvectors remained unspecified. This reduced the *g*-spectrum to a degenerate neutral state which was identified as the neutral convective state of the fluid. The various matrices of Equation (1) acquired the following forms (hereafter to emphasize the neutral aspect of the displacements the index *g* is replaced by *n*):

$$T^0=\left[\begin{array}{c|c}0 & 0\\ \hline 0 & T^0_{pp}\end{array}\right], \tag{3a}$$

$$S^0=\left[\begin{array}{c|c}S^0_{nn} & 0\\ \hline 0 & S^0_{pp}\end{array}\right], \tag{3b}$$

$$Z^0=\left[\begin{array}{c|c}Z^0_{nn} & 0\\ \hline 0 & Z^0_{pp}\end{array}\right], \tag{3c}$$

and

$$E^0=\left[\begin{array}{c|c}0 & 0\\ \hline 0 & E^0_{p}\end{array}\right],\qquad E^0_p=\text{diagonal.} \tag{3d}$$

Equation (1) reduced to its *p*-component

$$T^0_{pp}Z^0_{pp}=S^0_{pp}Z^0_{pp}E^0_p\,. \tag{4}$$

This completes the summary of the background material on unperturbed systems. The effect of a perturbation on Equation (1) is considered in the following section.

III. The Perturbed System

Let the adiabatic fluid discussed above be subjected to a perturbing force. Let Equation (1) take the more general form

$$TZ=SZE \tag{5}$$

with the following partitioned components

$$\left[\begin{array}{c|c}T_{nn}Z_{nn}+T_{np}Z_{pn} & T_{nn}Z_{np}+T_{np}Z_{pp}\\ \hline T_{pn}Z_{nn}+T_{pp}Z_{pn} & T_{pn}Z_{np}+T_{pp}Z_{pp}\end{array}\right] \tag{5a}$$

$$=\left[\begin{array}{c|c}(S_{nn}Z_{nn}+S_{np}Z_{pn})E_n & (S_{nn}Z_{np}+S_{np}Z_{pp})E_p\\ \hline (S_{pn}Z_{nn}+S_{pp}Z_{pn})E_n & (S_{pn}Z_{np}+S_{pp}Z_{pp})E_p\end{array}\right]. \tag{5a}$$

On the assumption of small perturbations, and in view of the particular form of Equations (3), we observe the following order-of-magnitude relations among the elements of various matrices:

$$T_{nn}, T_{np}, T_{pn}\ll T_{pp}\,, \tag{6a}$$

$$S_{np}, S_{pn}\ll S_{nn}, S_{pp}\,, \tag{6b}$$

$$Z_{np}, Z_{pn}\ll Z_{nn}, Z_{pp} \tag{6c}$$

and

$$E_n\ll E_p\,. \tag{6d}$$

The *nn*- and the *pn*-blocks of Equation (5a) do not contain zero order terms. Retaining only the first-order terms of the latter blocks gives

$$T_{nn}Z_{nn}=S_{nn}Z_{nn}E_n \tag{7}$$

and

$$T_{pn}Z_{nn}+T_{pp}Z_{pn}=0\,. \tag{8}$$

We emphasize that T_{nn} in Equation (7) is a pure perturbation term. Equation (7) alone determines the eigenfrequencies of the perturbation spectrum and uniquely specifies the corresponding eigenvectors Z_{nn}. This determination of Z_{nn}, a privilege not shared by Z^0_{nn} of Equation (3c), is what has been referred to as the removal of degeneracy from the neutral state. Equation (7) will constitute the core of the remainder of the paper. Having found Z_{nn}, Equation (8), now a set of linear inhomogeneous equations, will determine the matrix Z_{pn}. We recall that Z_{pn} is the correction to the no longer degenerate eigenvectors Z_{nn}. It expresses the perturbation-induced effects of the *p*-modes on the neutral states. From Equation (8) it is apparent that Z_{pn}, consistant with the anticipation in Equation (6c), is much smaller than Z_{nn}. We therefore conclude:

Under the effect of a perturbation a new spectrum of oscillations evolves from the neutral states of the fluid. The eigenfrequencies and the eigenvectors of the spectrum are specified by the perturbation field alone. There is a coupling between the new modes and the everpresent *p*-modes. This, however, is small and is proportional to the perturbation.

The *pp*- and the *np*-blocks of Equation (5a) are concerned with the *p*-modes and the effects of the perturbing forces on them. They will not be discussed further.

IV. The T- and S- Matrices

Let ζ^i_n; $i=1,2,\ldots$, be a set of basis-vectors belonging to the space of the neutral displacements of the unperturbed fluid. By definition

$$T^{ij}_{nn}=\int \zeta^i_n\cdot\mathscr{T}\zeta^j_n dV \tag{9}$$

and

$$S^{ij}_{nn}=\int \zeta^i_n\cdot\varrho\zeta^j_n dV\,, \tag{10}$$

where $\mathscr{T}$ is the differential operator governing the Lagrangian displacements and ϱ is the density of the fluid. To obtain $\mathscr{T}$ let the hydrostatic equilibrium be given by

$$-\nabla p+\varrho\nabla\Omega+\lambda \boldsymbol{F}=0\,, \tag{11}$$

where, it will be assumed, the perturbing force $\lambda\boldsymbol{F}$ is small compared with the other terms. The equation governing the Lagrangian displacement, $\xi(r)\exp(i\sqrt{\varepsilon}t)$, is

$$\mathscr{T}\xi=\varepsilon\varrho\xi\,, \tag{12}$$

where

$$\mathscr{T}\xi=\nabla(\delta p)-\delta\varrho\nabla\Omega-\varrho\nabla(\delta\Omega)-\lambda\delta\boldsymbol{F}\,, \tag{13}$$

$$\delta p=-\gamma p\nabla\cdot\xi-\xi\cdot\nabla p\,, \tag{13a}$$

$$\delta\varrho=-\varrho\nabla\cdot\xi-\xi\cdot\nabla\varrho\,, \tag{13b}$$

$$\delta\Omega=G\int|\chi-\chi'|^{-1}\delta\varrho(X')dV', \tag{13c}$$

and

$$\delta\boldsymbol{F}=\mathscr{F}\xi\,. \tag{13d}$$

The last equation is a symbolic representation of the Eulerian change in $\boldsymbol{F}$. To the first order in λ let

$$p=p_0+\lambda p_1\,,\qquad \varrho=\varrho_0+\lambda\varrho_1\,,\qquad \Omega=\Omega_0+\lambda\Omega_1\,. \tag{14}$$

The functions p, ϱ, Ω, their unperturbed values p_0, ϱ_0, Ω_0 and their perturbation terms p_1, ϱ_1 and Ω_1 are, in principle to be obtained from a solution of the equilibrium Equation (11). Their determination is a problem separate from the oscillation analysis of the fluid and will not be pursued here.

Substituting Equation (14) in Equations (13a–d), then substituting the results in Equation (13) gives

$$\mathscr{T}\xi=\mathscr{T}_0\xi+\lambda\mathscr{T}_1\xi\,, \tag{15}$$

where the explicit expressions for $\mathscr{T}_0$ and $\mathscr{T}_1$ are

$$\mathscr{T}_0\xi=-\nabla(\gamma p_0\nabla\cdot\xi+\nabla p_0\cdot\xi)+\nabla\Omega_0\nabla\cdot(\varrho_0\xi)+G\varrho_0\nabla\int|\chi-\chi'|^{-1}\nabla'\cdot(\varrho_0\xi)'dV' \tag{15a}$$

and

$$\mathscr{T}_1\xi=-\nabla(\gamma p_1\nabla\cdot\xi+\nabla p_1\cdot\xi)+\nabla\Omega_0\nabla\cdot(\varrho_1\xi)+G\varrho_0\nabla\int|\chi-\chi'|^{-1}\nabla'\cdot(\varrho_1\xi)'dV'+\nabla\Omega_1\nabla\cdot(\varrho_0\xi)+G\varrho_1\nabla\int|\chi-\chi'|^{-1}\nabla'\cdot(\varrho_0\xi)'dV'-\mathscr{F}\xi\,. \tag{15b}$$

The basis vectors ζ_n^i of Equations (9) and (10) belong to the space of neutral displacements of the unperturbed fluid. The Eulerian changes δp_0, $\delta\varrho_0$ and $\delta\Omega_0$ corresponding to these vectors are all zero. Consequently

$$\mathscr{T}_0\zeta_n^i=0\,. \tag{16}$$

Equations (9), (15) and (16), with λ suppressed, give

$$T_{nn}^{ij}=\int\zeta_n^i\cdot\mathscr{T}_1\zeta_n^j dV\,. \tag{17}$$

Once again let us observe that T_{nn} is determined entirely by the perturbing force and tends to zero as the latter. The expression for S_{nn} remains unchanged except that it will be approximated by its unperturbed value S_{nn}^0.

V. The Neutral Basis-Vectors ζ_n^i

Owing to its shear-free nature a toroidal displacement of a fluid is always a neutral displacement. Let $\zeta_t^i;\, i=1,2,\ldots,$ be a set of basis vectors spanning the subspace of the toroidal displacements. In an adiabatic fluid for which the structural gradients of p and ϱ are equal to their adiabatic gradients, convective motions (or equivalently the g-modes) are also neutral displacements. Let $\zeta_c^i;\, i=1,2,\ldots,$ be a set of basis vectors spanning the subspace of the neutral convective displacements. In addition to these a solid-body translation of the fluid is also a neutral displacement. A single vector is sufficient to describe the translation. For brevity, however, the latter will be included in the set of ζ_c-vectors. Solid-body rotations of the fluid which also come under neutral displacements are a special case of toroidal motions and are already accounted for in the ζ_t-set. The displacements just discussed seem to comprise all possible neutral motions. We therefore proceed with the assumption that the basis vectors $\zeta_n^i:(\zeta_c^j,\zeta_t^k)$, where ζ_c^j also comprises the solid-body translations, are complete and span the whole space of the neutral displacements of an adiabatic fluid.

Division of neutral displacements into convective and toroidal displacements entails a corresponding partitioning of the matrices of Equation (7). Thus

$$A=\left[\begin{array}{c|c}A_{cc} & A_{ct}\\ \hline A_{tc} & A_{tt}\end{array}\right],\qquad A=\boldsymbol{T}_{nn},\boldsymbol{S}_{nn}^0,\boldsymbol{Z}_{nn},\boldsymbol{E}_n\,. \tag{18}$$

The subspace of the convective modes is orthogonal to the subspace of the toroidal modes in that the S-matrix is block-diagonal:

$$\boldsymbol{S}_{nn}^0=\left[\begin{array}{c|c}\boldsymbol{S}_{cc}^0 & 0\\ \hline 0 & \boldsymbol{S}_{tt}^0\end{array}\right]. \tag{19}$$

If the nature of the perturbing force is such that the T-matrix of Equation (17) is also block-diagonal in exactly in the same manner as $\boldsymbol{S}_{nn}^0$, then there is no coupling between the convective and the toroidal modes. Each displacement can then be analized separately. The latter ruling also applies to a simultaneous and conformable subpartitioning of any of the blocks of $\boldsymbol{T}$ and $\boldsymbol{S}$ into block-diagonalized forms. In Section VI we will have occasions to utilize this property. Most perturbing forces, however, e.g. magnetic field, rotation and tidal forces, couple the convective and toroidal motions.

VI. Pure Hydromagnetic Spectra of Adiabatic Fluids

A magnetic field pervading an otherwise adiabatic fluid will be capable of removing the degeneracy of the neutral state and give rise to a sequence of oscillations primarily determined by magnetic forces. Such oscillations, should they develop, will have very long periods, of the order of Alfvèn-crossing times. For the sun and for global magnetic fields of a few thousand Gauss (the order of magnitude of strong sunspot fields) these periods are as large as the period of the solar magnetic cycle.

Owing to the complexity of the $\mathscr{T}_1$ operator [Eq. (15b)] elaboration of even the simplest conceivable example requires enormous amount of mathematical manipulations and numerical computations. As a very simple example we restrict ourselves to force-free-magnetic fields. Let the perturbing force of Equation (11) be

$$\boldsymbol{F}=-\frac{I}{4\pi}\boldsymbol{H}\times(\nabla\times\boldsymbol{H})=0\,. \tag{20}$$

The hydrostatic equilibrium will not be altered by this force-free field. The perturbation terms p_1, ϱ_1, Ω_1 and all their derivitives will be identically zero. Equations (17) and (15b) will then give (suppressing the index n)

$$T^{ij}=-\int\zeta^i\cdot\mathscr{F}\zeta^j dV, \tag{21}$$ *

where, ζ^i and ζ^j are members of the sets of convective and/or toroidal basis vectors, ζ_c and/or ζ_t, respectively,

$$\mathscr{F}\xi=-\frac{I}{4\pi}[\delta\boldsymbol{H}\times(\nabla\times\boldsymbol{H})+\boldsymbol{H}\times(\nabla\times\delta\boldsymbol{H})] \tag{22a}$$

and

$$\delta\boldsymbol{H}=\nabla\times(\xi\times\boldsymbol{H})\,. \tag{22b}$$

Equation (22b), giving the Eulerian variations of the magnetic field, assumes infinite conductivity and expresses the frozen-in condition of the field. Substituting Equations (22) in Equation (21), performing some integrations by parts and eliminating the surface integrals by virtue of the fact that a force-free field has no component normal to the boundary surface, give

$$\begin{aligned}T_{ij}=&\frac{1}{4\pi}\int H^2(\nabla\cdot\zeta^i)(\nabla\cdot\zeta^j)dV\\&+\frac{1}{4\pi}\int\{(\boldsymbol{H}\cdot\nabla)\zeta^i\}\cdot\{(\boldsymbol{H}\cdot\nabla)\zeta^j\}dV\\&-\frac{1}{4\pi}\int[\boldsymbol{H}\cdot\{(\boldsymbol{H}\cdot\nabla)\zeta^i\}(\nabla\cdot\zeta^j)\\&+\boldsymbol{H}\cdot\{(\boldsymbol{H}\cdot\nabla)\zeta^j\}(\nabla\cdot\zeta^i)]dV\\&+\frac{1}{8\pi}\int[\zeta^i\cdot\{\zeta^j\cdot\nabla[(\boldsymbol{H}\cdot\nabla)\boldsymbol{H}]\}\\&+\zeta^j\cdot\{\zeta^i\cdot\nabla[(\boldsymbol{H}\cdot\nabla)\boldsymbol{H}]\}]dV\\&+\frac{1}{8\pi}\int[\{(\zeta^i\cdot\nabla)H^2\}(\nabla\cdot\zeta^j)\\&+\{(\zeta^j\cdot\nabla)H^2\}(\nabla\cdot\zeta^i)]dV.\end{aligned} \tag{23}$$

For manipulations leading to Equation (23), except for indexing, one may consult Kovetz (1966), Singh and Tandon (1969) and/or Grover et al. (1973). Kovetz' paper should be noted for its careful analysis of the boundary conditions in the presence of a magnetic field of a quite general nature. A reference to it will reveal how the surface integrals coming from integrations by parts vanish in case the field is force-free. Kovetz also extends the variational principle of Chandrasekhar (1964) to magnetized fluids. The symmetry of the matrix of Equation (23), which is at the root of the present variational analysis, is reminiscent of this general variational principle.

An axially symmetric force-free field, consisting of totoidal and poloidal components can be constructed as follows (see Ferraro and Plumpton, 1966)

$$\begin{aligned}\boldsymbol{H}=&\frac{n(n+1)}{\alpha r}Z_n(\alpha r)Y_n(\theta)\,,\\&\frac{1}{\alpha}\left(\frac{d}{dr}+\frac{1}{r}\right)Z_n(\alpha r)\frac{dY_n}{d\theta}\,,\\&Z_n(\alpha r)\frac{dY_n}{d\theta}\,,\end{aligned} \tag{24}$$

where,

$$Z_n(x)=\left(\frac{\pi}{2x}\right)^{1/2}J_{n+1/2}(x)\,, \tag{25}$$

$J_{n+1/2}$ is the Bessel function of order $n+\frac{1}{2}$ and $Y_n(\theta)$ is the spherical harmonic of order n. In order for the normal component of the field to vanish on the sphere of radius R one must have

$$J_{n+1/2}(\alpha R)=0\,. \tag{25a}$$

The latter equation determines the value α.

The basis vectors ζ_c and ζ_t can also be given a spherical harmonic expansion. Most generally one may write

$$\begin{aligned}\zeta_c^{ilm}=&\frac{1}{r^2}\psi^{ilm}Y_l^m\,,\quad\frac{1}{l(l+1)}\frac{1}{dr}\frac{d\chi^{ilm}}{dr}\frac{\partial Y_l^m}{\partial\theta}\,,\\&\frac{1}{l(l+1)}\frac{1}{r}\frac{d\chi^{ilm}}{dr}\frac{1}{\sin\theta}\frac{\partial Y_l^m}{\partial\varphi}\end{aligned} \tag{26}$$

and

$$\zeta_t^{jkp}=0\,,\quad\frac{1}{r^2}\phi^{jkp}\frac{1}{\sin\theta}\frac{\partial Y_k^p}{\partial\varphi}\,,\quad-\frac{1}{r^2}\phi^{jkp}\frac{\partial Y_k^p}{\partial\theta}\,, \tag{27}$$

where ψ, χ and ϕ are scalar functions and will be elaborated on later. The next step is to substitute Equations (24), (26) and (27) in Equation (23) and to carry out the angular integrations over θ and φ. We observe that the matrix of Equation (23) is quadratic in $\boldsymbol{H}$ and quadratic in the pair of vectors (ζ^i, ζ^j). Each term of the matrix contains an angular integration over four spherical harmonics. The following conclusions follow from a parity analysis and the integrals.

a) Integrals over φ vanish unless the azimuthal harmonic numbers m and p coming from Equations (26) and (27) are the same. Therefore the magnetic field, whether axial or not, couples displacements of the same azimuthal number, m. According to the remark at the end of section V, a given value of m, regardless of the

* Note added in proof see page 346

value of the principle harmonic number, l, should be analyzed independently from other values of m. Only axially symmetric displacements i.e. $m=0$, are considered in this paper.

b) Integrals over θ have the parity $2n+k+l$. They will vanish unless $k+l$ coming from Equations (26) and (27) is even. Therefore, the axially symmetric displacements of odd parity, i.e. k, $l=1,3,\ldots$, will be coupled together and those of even parity, i.e. k, $l=2,4,\ldots$, will be coupled together. There is no coupling between the displacements of odd and of even parities. There are no neutral displacements of either convective or toroidal type to correspond to k, $l=0$.

The indices i and j of Equations (26) and (27) should in principle run over an infinite range of values. In the variational approximation of this paper, however, only one value of these indices is allowed. Provided the single functions ψ and ϕ of Equations (26) and (27) are reasonable representations of the actual displacements, the latter choice is a legitimate variational approximation. Let us summarize the discussion. Convective basis vectors will have the form

$$\zeta_c^l = \frac{1}{r^2}\psi^l(r)Y_l(\theta)\,, \quad \frac{1}{l(l+1)}\frac{1}{dr}\frac{d\chi^l(r)}{dr}\frac{dY_l}{d\theta}\,, \quad 0. \tag{28}$$

The unperturbed fluid is to have adiabatic gradients and the Eulerian variations of pressure corresponding to ζ_c^l are to vanish. These requirements impose the following restriction:

$$\frac{d\chi^l}{dr} = \frac{d\psi^l}{dr} + \frac{1}{\varrho}\frac{d\varrho}{dr}\psi^l\,. \tag{28a}$$

Similarly the toroidal vectors will have the form:

$$\zeta_t^k = 0, 0, -\frac{\phi^k(r)}{r^2}\frac{dY_k}{d\theta}\,. \tag{29}$$

Single scalar functions ψ^l and ϕ^k will specify each of the convective and toroidal vectors, respectively.

VII. Ansatz for ψ^l and ϕ^l

It is shown by Hurley et al. (1966) that ψ^l should behave as r^{l+1} at the origin. It was also shown in paper I that ψ^l should vanish as $|\boldsymbol{R}-\boldsymbol{r}|$ at the surface. The magnetic field entering Equation (23) is given in terms of Bessel functions which vanish at the surface, cf. Equations (24) and (25). In view of these considerations ψ^l was chosen as follows

$$\psi^l = \frac{(2n+1)!}{2^n n!}\frac{Z_n(\alpha r)}{(\alpha r)^n}r^{l+1}\,. \tag{30}$$

The numerical factor is introduced to ensure the order-of-magnitude consistancy in the computations.

The expressions below for ϕ^l are empirical. They are chosen from a rather large number of experimental computations on the basis of good first estimates for eigenfrequencies and by the subsequent convergence of the results. The following observation, however, provided guidance in the course of trial computations.

a) ϕ^l For Odd Harmonics—We recall the solid body translation of the fluid was a member of ζ_c displacements. This is actually a p-mode of $l=1$ which happens to be neutral and for which $\psi^{tr}=\chi^{tr}=r^2$. It was shown in paper I that in an adiabatic fluid all p-modes are orthogonal to the convective-displacements of Equation (28). The non-diagonal elements of the S-matrix corresponding to a pair of convective- and p-displacements vanish. However, to choose a displacement orthogonal to the solid-body translation of the fluid is simply to require that the displacement in question should not impart a net momentum to the system. Thus

$$S_{cc}^{l,tr} = \int_0^R \varrho[\psi^l + r\chi^{l\prime}]dr = 0\,, \quad l=1\,. \tag{31}$$

Equation (31) is indeed satisfied if ψ and χ are related by Equation (28a).

A parallel argument can be advanced with regard to the toroidal functions, ϕ^l. A solid body rotation of the fluid is member of ζ_t-displacements. It corresponds to $l=1$ and $\phi^{\text{rot}}=r^3$. Other values of $\phi^l(l=1)$ can be chosen orthogonal to ϕ^{rot}. The condition is that the ϕ^l in question should not impart a net angular momentum to the system. Thus

$$S_{tt}^{l,\text{rot}} = -\int_0^R \varrho\phi^l r dr = 0\,, \quad l=1\,. \tag{32}$$

There is no evidence as to how ϕ^l should behave at the center. Our numerical calculations, however, indicated faster convergence if ϕ^l behaved the same as ψ^l i.e. as r^{l+1}. On requiring this and also requiring ϕ^l to remain finite at the surface and comparing the particular forms of Equations (31) and (32) one arrives at the following expression:

$$\phi^l = \psi^l + \frac{1}{2}r\frac{d\chi^l}{dr} \quad l=1,3,5,\ldots\,. \tag{33}$$

Although Equation (33) is derived for $l=1$, it was found suitable also for other odd values of l.

b) ϕ^l For Even Harmonics—The guideline developed in the odd case no longer applies. From a large number of functions behaving as r^{l+1} at the center and remaining finite on the surface the following was chosen:

$$\phi^l = \frac{1}{l(l+1)}r\frac{d\chi^l}{dr}\,, \quad l=2,4,\ldots\,. \tag{34}$$

VIII. Computational Procedure

The matrix elements T_{cc}^{kl}, T_{ct}^{kl} and T_{tt}^{kl} were calculated by substitution of the magnetic field of Equations (24), (25) and the appropriate vectors ζ_c^k and/or ζ_t^l in Equations (23).

Table 1. Eigenvalues and eigenvectors of pure hydromagnetic oscillations of a convectively neutral fluid. Eigenvalues are displayed in lines marked by an asterisk. Columns following the eigenvalues are the corresponding eigenvectors. The convective and the toroidal components of the eigenvectors are marked by "C" and "T", respectively. The unit of eigenvalues is the ratio of the total magnetic energy to the moment of inertia of the system

$N=1$, Uneven harmonics:					
*	0.418406 +00				
C	0.100000 +01				
*	0.409282 +00	0.165124 +01			
C	0.100000 +01	−0.478386 −01			
T	0.154712 +00	0.100000 +01			
*	0.407409 +00	0.738877 +00	0.377757 +01		
C	0.100000 +01	0.450205 −01	−0.779420 −02		
T	0.248057 +00	−0.480580 +00	0.207989 +00		
C	−0.261019 +00	0.100000 +01	0.100000 +01		
*	0.406296 +00	0.727131 +00	0.370986 +01	0.815705 +01	
C	0.100000 +01	0.408131 −01	−0.893291 −02	0.135675 −02	
T	0.233435 +00	−0.473535 +00	0.213310 +00	0.730700 −02	
C	−0.217314 +00	0.100000 +01	0.100000 +01	0.773421 −01	
T	−0.113919 +00	−0.117628 +00	−0.242074 +00	0.100000 +01	
*	0.406032 +00	0.725569 +01	0.361707 +01	0.437520 +01	0.100665 +02
C	0.100000 +01	0.400449 −01	0.988174 −02	0.129921 −03	−0.934671 −03
T	0.231011 +00	−0.472587 +00	−0.202547 +00	0.292290 −01	−0.383562 −02
C	−0.209907 +00	0.100000 +01	−0.917097 +00	0.168936 +00	−0.479858 −01
T	−0.139188 +00	−0.137928 +00	0.574763 +00	0.382332 +00	−0.825525 +00
C	−0.117628 +00	−0.940668 −01	0.100000 +01	0.100000 +01	0.100000 +01
$N=1$, Even harmonics:					
*	0.207413 +01				
C	0.100000 +01				
*	0.205897 +01	0.351999 +01			
C	0.100000 +01	0.768565 −01			
T	−0.136411 +00	0.100000 +01			
*	0.205398 +01	0.325654 +01	0.436817 +01		
C	0.100000 +01	0.734586 −01	−0.417766 −02		
T	−0.158772 +00	0.660260 +00	−0.207620 +00		
C	−0.185641 +00	0.100000 +01	0.100000 +01		
*	0.205293 +01	0.323440 +01	0.410432 +01	0.154172 +02	
C	0.100000 +01	0.823639 −01	0.170411 −02	−0.293386 −02	
T	−0.149869 +00	0.793929 +00	−0.170292 +00	−0.279452 −01	
C	−0.201634 +00	0.100000 +01	0.100000 +01	0.830663 −01	
T	0.491729 −01	−0.165423 +00	−0.239370 +00	0.100000 +01	

The convective vectors were taken from Equations (28) and (30). The density distribution to be used in Equation (28a) was that of polytrope 3/2, the adiabatic fluid corresponding to the ratio of specific heats 5/3. The toroidal vectors were taken from Equations (29) and (33) or (34). After eliminating the derivatives of spherical harmonics, each angular term consisted of an integration over four Legendre polynomials. The latter were calcalculated by the following formula.

$$\frac{1}{2}\int_{-1}^{1} P_i(x)P_j(x)P_k(x)P_l(x)dx = \sum_{n=|k-l|}^{k+l} (2n+1)\begin{pmatrix} i & j & n \\ 0 & 0 & 0 \end{pmatrix}^2 \begin{pmatrix} k & l & n \\ 0 & 0 & 0 \end{pmatrix}^2, \tag{35}$$

where $\begin{pmatrix} j_1 & j_2 & j_3 \\ 0 & 0 & 0 \end{pmatrix}$ is a 3-j symbol. Among other properties we recall that 3-j symbols are zero unless j_1, j_2, j_3 satisfy the triangle condition $j_k+j_l-j_m \geqq 0$ for any permutation (k, l, m) of (1,2,3). The statements given after Equation (27) can be verfied by employing the latter property of the 3-j symbols. The numerical expression is:

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ 0 & 0 & 0 \end{pmatrix} = (-1)^{J/2}\left[\frac{(J-2j_1)!(J-2j_2)!(J-2j_3)!}{(J+1)!}\right]^{1/2} \cdot \frac{(\frac{1}{2}J)!}{(\frac{1}{2}J-j_1)!(\frac{1}{2}J-j_2)!(\frac{1}{2}J-j_3)!}, \quad J=j_1+j_2+j_3. \tag{36}$$

For further reading on 3-j symbols see, for example, Edmonds (1957). After carrying out angular integrations, integrations over r were carried out numerically.

Having obtained the T- and S-matrix, solutions of Equation (7) were attempted. In variational calculations from one to at most six variational parameters, from the collection of convective and toroidal terms, were considered. The eigenvalues, i.e. the squares of the eigenfrequencies were expressed in units of the ratio of the total magnetic energy to the moment of inertia of the fluid. The latter are

$$\frac{1}{8\pi}\int H^2 dV = \frac{n(n+1)}{8\pi} Z_{n-1}^2(z_1) H^2 R^3 \tag{37a}$$

$$\int \varrho r^2 dV = 0.3069 MR^2 \,, \tag{37b}$$

respectively, where z_1 is the first zero for Z_n, H is the physical amplitude of the magnetic field, and R and M are the physical radius and the physical mass of the convecting fluid, respectively. All calculations of basis vectors, matrix elements, eigenvectors, etc..., are done with the radius of the fluid normalized to unity. This means that the variable r, throughout the paper should be interpreted as r/R. Particularly in the interpretation of the eigenvectors the adoption of unit radius should be kept in mind.

Eigenvalues and Eigenvectors of $n=1$ (n being the harmonic number in the expansion of the magnetic field) are given in the accompanying table. Eigenvalues are displayed in lines marked by an asterisk. The column following an eigenvalue is the corresponding eigenvector. Convective and toroidal components of the vectors are marked explicitly. Let us remember that an eigendisplacement, ξ, is to be calculated from

$$\xi = \sum_l a_c^l \zeta_c^l + a_t^l \zeta_t^l \,, \tag{38}$$

where a_c^l and a_t^l are the convective and the toroidal components of the corresponding eigenvector.

Example—From the table, the column eigenvector of the lowest uneven mode in the third approximation is (1, 0.248, −0.261). The corresponding eigendisplacement will be

$$\xi(r,\theta) = \zeta_c^1(r,\theta) + 0.248\,\zeta_t^1(r,\theta) - 0.261\,\zeta_c^3(r,\theta) \tag{38a}$$

where, ζ_c^1 and ζ_c^3 are to be obtained from Equations (28) and (30) and ζ_t^1 from Equations (29) and (33).

IX. Remarks on Long-Period Aspects of the Oscillations and the Displacement Patterns

Strictly speaking, the numerical results of this paper are pertinent to a fluid in a globally neutral convective state and pervaded by a force-free magnetic field. The convective envelope of the sun with a mass of about 0.002 $M_\odot$ and a thickness of 0.15 $R_\odot$ may fulfill the assumption of neutral stability and is presumably the seat of some large scale magnetic fields. There is the temptation to contemplate that at least the gross features of the present calculations are of relevance to the magnetic periodicity of the sun. In no way, however, should the remark be interpreted to imply a theory of solar magnetic activity or the sunspots.

From the table the eigenvalues of the lowest odd and even modes are respectively, 0.406 and 2.053, in the units of Equations (37). The corresponding periods in a physical unit are

$$P^{\text{odd}}(\text{yrs}) = 4.78\;10^5 \left(\frac{M/M_\odot}{R/R_\odot}\right)^{1/2} \frac{1}{H(\text{Gauss})} \tag{39a}$$

and

$$P^{\text{even}}(\text{yrs}) = 2.13\;10^5 \left(\frac{M/M_\odot}{R/R_\odot}\right)^{1/2} \frac{1}{H(\text{Gauss})} \tag{39b}$$

For the mass and the radius of the solar convective envelope and for large scale magnetic fields of a few thousand gauss, typical of local sunspot fields, these periods fall well within the 22- and 11-year solar magnetic cycles.

The displacement vectors corresponding to the lowest uneven mode in the second approximation is

$$\xi_r^{\text{odd}} = \sqrt{\frac{3}{4\pi}}\frac{\psi^1(r/R)}{(r/R)^2}\cos\theta \,,$$

$$\psi^1(x) = \frac{3}{2z_1} x Z(z_1 x)\,, \qquad Z_1(z_1) = 0\,,$$

$$\xi_\theta^{\text{odd}} = -\frac{1}{2}\sqrt{\frac{3}{4\pi}}\frac{\chi^{1\prime}(r/R)}{(r/R)}\sin\theta \,,$$

$$\chi^{1\prime}(x) = \frac{d\psi^1}{dx} + \frac{1}{\varrho}\frac{d\varrho}{dx}\psi^1 \,,$$

$$\xi_\varphi^{\text{odd}} = 0.155\sqrt{\frac{3}{4\pi}}\frac{\left[\psi^1(r/R) + \frac{1}{2}\frac{r}{R}\chi^{1\prime}(r/R)\right]}{(r/R)^2}\sin\theta \,. \tag{40}$$

The r-component has different signs in different hemispheres. The θ- and φ-components have the same signs in both hemispheres. The φ-component is a toroidal shear-free displacement and is about 15% of the r- and θ-components. On the surface the r-component vanishes while θ- and φ-components tend to a finite limit.

The displacement vector corresponding to the lowest even mode in the second approximation is

$$\xi_r^{\text{even}} = \frac{1}{2}\sqrt{\frac{5}{4\pi}}\frac{\psi^2(r/R)}{(r/R)^2}(3\cos^2\theta - 1)\,,$$

$$\psi^2(x) = \frac{3}{2z_1} x^3 Z_1(z_1 x)\,,$$

$$\xi_\theta^{\text{even}} = -\frac{1}{2}\sqrt{\frac{5}{4\pi}}\frac{\chi^{2\prime}(r/R)}{(r/R)}\cos\theta\sin\theta \,,$$

$$\chi^{2\prime}(x) = \frac{d\psi^2}{dx} + \frac{1}{\varrho}\frac{d\varphi}{dx}\psi^2 \,,$$

$$\xi_\varphi^{\text{even}} = -\frac{0.136}{2}\sqrt{\frac{5}{4\pi}}\frac{\chi^{2\prime}(r/R)}{(r/R)}\cos\theta\sin\theta \,. \tag{41}$$

The r-component has the same sign in both hemispheres while the θ- and φ-components reverse the sign from one hemisphere to another. The toroidal φ-component is again much smaller than the other ones. Along the z-axis and in the equatorial plane the displacement is purely radial. Non-radial components are most pronounced in the $\theta=45°$-directions. The surface behavior is the same as for the odd case.

X. Convective Stability of Perturbed Configurations

The reasoning of Sections II and III has shown that a conservative perturbing force removes the degeneracy of the combined convective and toroidal neutral state of an adiabatic fluid. The new sequence of normal modes which developes is solely characterized by the restoring forces induced by the perturbation. Due to the conservative nature of the perturbing force, the corresponding frequencies are either real or imaginary but never complex. And due to the weak nature of the perturbation they are much smaller than the acoustic frequencies of the fluid. For example the force-free magnetic field gives rise to purely hydromagnetic oscillations. The frequencies as judged from the present calculations form an ascending sequence. The corresponding periods are of the order of Alfvèn crossing-times.

The picture presented above is intimately related to the question of the convective stability in the presence of a perturbing force. Should the perturbation spectrum be a sequence of stable oscillations then the perturbation has stabilized an otherwise neutral fluid, and vice versa. On making the structural gradients of the density and pressure steeper than the adiabatic ones the fluid will continue to remain stable until the lowest eigenstate induced by combined superadiabatic and perturbation forces becomes unstable. A precise statement and a proof of the assertion just made, is attempted in a subsequent paper. An elucidation, however, is given below.

Let the fluid be slightly non-adiabatic. In the absence of a perturbation it will develop a g- or equivalently a convective-sequence of normal modes. Let there be a one to one correspondence between these g-states and the perturbation-states. That is, assume that the eigendisplacements of the non-adiabatic unperturbed fluid, as non-adiabaticity tends to zero, have the same limits as the eigendisplacements of the adiabatic but perturbed fluid, as the perturbation tends to zero. Incidentally this is not the case for the force-free magnetic field of this paper, and that is why the analysis of the general case is more involved. If the fluid is both slightly non-adiabatic and slightly perturbed the corresponding eigenvalues will simply add together. Let ε_g and ε_{per} be the lowest g-eigenvalue and the lowest perturbation eigenvalue, respectively. The fluid will be convectively stable if

$$\varepsilon_g+\varepsilon_{\text{per}}\geqq 0\,. \tag{42}$$

It was shown in paper I that as the Schwarzschild discriminant $A=\nabla_{\text{ad}}\varrho-\nabla\varrho$ tends to zero all g-eigenvalues tend to zero proportionally to A. Define a as follows

$$A=a\nabla\varrho=\left[\left(\frac{\partial\ln\varrho}{\partial\ln p}\right)_{\text{ad}}\Big/\left(\frac{\partial\ln\varrho}{\partial\ln p}\right)-1\right]\nabla\varrho\,. \tag{43}$$

By the statement just made

$$\varepsilon_g=\left(\frac{d\varepsilon_g}{da}\right)_0 a \quad \text{as} \quad a\to 0\,. \tag{44}$$

The g-eigenvalues of subadiabatic fluids are positive and those of superadiabatic ones are negative. Therefore, ε_g and a are always of opposite signs and $(d\varepsilon_g/da)_0$ is negative. Substituting Equation (44) in Equation (42) the condition for convective stability becomes

$$\left(\frac{\partial\ln\varrho}{\partial\ln p}\right)_{\text{ad}}\Big/\left(\frac{\partial\ln\varrho}{\partial\ln p}\right)-1\leqq\frac{\varepsilon_{\text{per}}}{-(d\varepsilon_g/da)_0}\,. \tag{45}$$

The criterion (45), in agreement with the conventional mode of thinking, corrects the Schwarzschild criterion for the effect of the perturbation. This has become possible on account of a crucial assumption that in the limit of vanishing non-adiabaticity the perturbation does not cause a mixing of the convective modes (or in the language of linear vector spaces, a rotation of the convective eigenvectors). Indeed the assumption is latent in any stability study of perturbed configurations as a local phenomenon and by analyzing the buoyancy forces. The consequences of abandoning the assumption are far reaching. They argue in favor of global stability criteria in terms of the integral properties of the fluid in contrast with local stability conditions in terms of the local structure of the fluid. Further analysis of this aspect of the problem is planned.

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Note added in proof. Since the submission of this paper the author has been able to establish the following result: As long as the condition of $p=p(\varrho)$ holds in the equilibrium state of fluid, contributions of the density and pressure perturbations, ϱ_1 and p_1 respectively, to the T_{nn}-matrix are identically zero. Equation (21) will then hold for any field configuration.