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Research Note

The Potentials for the *g*-, *p*-, and the Toroidal-modes of Self-gravitating Fluids*

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Summary. In a convectively neutral fluid the g-modes are derived from a vector potential and the p-modes from a scalar potential. In a convectively non-neutral fluid the two potentials are coupled. For small and moderate deviations from convective neutrality, however, the solenoidal character of the g-modes and the irrotational nature of the p-modes persist.

Key words: normal modes – pulsation

I. Introduction

In a study of convective stability of self-gravitating fluids Kaniel and Kovetz (1967) introduced a decomposition of the linear motions of the fluid into a toroidal component, an irrotational component, and a component $\boldsymbol{\eta}=\varrho^{-1}\nabla\times\boldsymbol{a}$, where ϱ is density of the fluid and $\boldsymbol{a}$ is a vector potential. Recently, Aizenman and Smeyers (1977) suggested to study the small oscillations of the fluid in terms of their potential fields. Their starting point was the Helmholtz theorem which enables one to decompose a vector field into a solenoidal and an irrotational component. Both of these papers, however, are silent about any possible connection between the various components of the motion and the well established g- and p-character of the modes. Here we elaborate on these concepts and show that the g-character exhibited by a linear motion is attributable to its solenoidal component and the p-character to its irrotational component. In Sect. II we study the structure of the vector space, H, of the linear motions of a continuous medium. We show that H can be divided into two orthogonal subspaces, H_Φ and H_A, where, the elements of H_Φ are derived from a scalar potential Φ and the elements of H_A are derived from a vector potential A. Pure g- and pure p-modes are only found in convectively neutral fluids. In Sect. III we demonstrate that the g- and p-modes of such a fluid are indeed exactly solenoidal and irrotational, respectively. In Sect. IV we develope an expression for the scalar potential.

II. The Structure of the Vector Space of Small Motions

The discussion below is general and applicable to any continuous medium whether a spherically symmetric normal star or not. The medium, however, is envisaged to have a free boundary on which the density ϱ and the pressure p exhibit a polytropic behaviour. That is, as the inward distance x from the surface tends to zero: $\varrho\to x^n$, $d\varrho/dx\to x^{n-1}$, $p\to x^{n+1}$, and $dp/dx\to x^n$, where n is an effective polytropic index at the surface.

A small displacement field of the medium, ξ, has finite normal component and finite divergence at the boundary. Thus,

$$\xi\cdot\boldsymbol{n}|_S=\text{finite},\tag{1a}$$

$$\nabla\cdot\xi|_S=\text{finite},\tag{1b}$$

where S denotes the boundary and $\boldsymbol{n}$ is a unit vector normal to it. Furthermore, ξ belongs to a vector space H in which the norm and the inner product are defined as

$$|\xi|^2=\int\varrho\xi^*\cdot\xi dv=\text{finite and positive};\quad \xi\in H,\tag{2a}$$

$$(\boldsymbol{\eta},\zeta)_\varrho=\int\varrho\boldsymbol{\eta}^*\cdot\zeta dv=\text{finite and real};\boldsymbol{\eta},\zeta\in H.\tag{2b}$$

The integration is over the volume of the fluid and ϱ is non negative.

According to Helmholtz, any continuous and once differentiable vector field is uniquely expressible in terms of a scalar and a vector potential. Let us apply this theorem to $\varrho\xi$:

$$\varrho\xi=-\nabla\Phi'+\nabla\times A',\ \nabla\cdot A'=0,\tag{3}$$

where Φ' and A' are solutions of

$$\nabla^2\Phi'=-\nabla\cdot(\varrho\xi),\tag{4a}$$

$$\nabla^2A'=-\nabla\times(\varrho\xi).\tag{4b}$$

The boundary conditions for Φ' and A' are obtained from Eqs. (1). To do this we observe that the potentials Φ' and A' in Eq. (3) are independent variables. For, arbitrarily many $\varrho\xi$ vectors can be constructed with the same $\nabla\cdot(\varrho\xi)$ [or $\nabla\times(\varrho\xi)$] but with different $\nabla\times(\varrho\xi)$ [or $\nabla\cdot(\varrho\xi)$]. In an actual fluid this can be acheived by arbitrary choices of the initial desplacement and the initial velocity fields. Therefore, the two components $\nabla\Phi'$, and $\nabla\times A'$ should independently satisfy the boundary conditions of Eqs. (1). Thus, as the distance x from the surface tends to zero, one obtains

$$\nabla\Phi'\cdot\boldsymbol{n}\to x^n,\tag{5a}$$

$$(\nabla\times A')\cdot\boldsymbol{n}\to x^{n+1}.\tag{5b}$$

A detailed derivation of Eqs. (5), when ξ is a normal mode of a fluid, can be found in Sobouti (1977, Sect. V).

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A Transformation of the Helmholtz Theorem

The Helmholtz decomposition of Eq. (3) is not compatible with the inner product of Eqs. (2). We propose the gauge transformation

$$\Phi' = \varrho\Phi + \Psi, \quad (6a)$$

$$A' = A + \Lambda, \quad \nabla \cdot \Lambda = 0. \quad (6b)$$

With the gauge condition of Eq. (8) below, Eq. (3) becomes

$$\varrho\xi = -\varrho\nabla\Phi + \nabla \times \Lambda, \quad (7)$$

$$\nabla \times \Lambda - \nabla\Psi - \nabla\varrho\Phi = 0. \quad (8)$$

The boundary conditions on Φ and Λ are obtained similarly to those on Φ' and A'. Thus,

$$\nabla\Phi \cdot \boldsymbol{n} \to \text{finite}, \quad (9a)$$

$$(\nabla \times \Lambda) \cdot \boldsymbol{n} \to x^{n+1}. \quad (9b)$$

To achieve the purpose we have to show that Eq. (8) can be solved for Λ and Ψ. Using Eq. (6a) to eliminate Φ from Eq. (8) and taking the divergence of the resulting equation gives

$$\nabla^2\Psi - \nabla \cdot (\varrho^{-1}\nabla\varrho\Psi) = -\nabla \cdot (\varrho^{-1}\nabla\varrho\Phi'). \quad (10a)$$

Taking the curl of Eq. (8) gives

$$\nabla^2\Lambda = -\nabla\Phi \times \nabla\varrho. \quad (10b)$$

The boundary conditions on Ψ and ∇, as read from Eqs. (6), (5), and (9), are

$$\nabla\Psi \cdot \boldsymbol{n} \to x^n \quad (10c)$$

$$(\nabla \times \Lambda) \cdot \boldsymbol{n} \to x^{n+1}. \quad (10d)$$

For brevity, we do not elaborate on the fact that Eqs. (10a) and (10c) can always be solved for Ψ. It is an easy matter to carry out a spherical harmonic expansion of Ψ and convince oneself that Eq. (10a) reduces to an ordinary second order differential equation with an appropriate boundary condition deduced from Eq. (10c). Having obtained Ψ one calculates Φ from Eq. (6a) and substitutes it in Eqs. (10b) and (10d) and solves for Λ. Equation (10b) is a standard vector Poisson equation.

To facilitate comparision between the decomposition of Eq. (7) and that of Kaniel and Kovetz, let us note that Eq. (7) was derived as direct consequence of the Helmholtz theorem and under much less restrictive conditions. The vector ξ in this equation is not required to belong to H.

The Subspaces of H

An immediate conclusion from Eq. (7) is the division of the vector space H into two orthogonal subspaces: One H_Φ with elements $\zeta_\Phi = -\nabla\Phi$, and the other H_A with elements $\zeta_A = \varrho^{-1}\nabla \times \Lambda$. The inner product of ζ_Φ and ζ_A is zero:

$$\int \varrho\zeta_\Phi^* \cdot \zeta_A = -\int \nabla\Phi^* \cdot (\nabla \times \Lambda) dv = -\int \Phi^* (\nabla \times \Lambda) \cdot dS + \int \Phi^* \nabla \cdot (\nabla \times \Lambda) dv = 0. \quad (11)$$

The Helmholtz decomposition of Eq. (3) and consequently that of Eq. (7) are unique. Therefore, the decomposition $H = H_\Phi + H_A$ is also unique.

The subspace H_A is, in turn, separable into two orthogonal subspaces. From Eqs. (3) and (6b) the vector potential A is divergence free. Two such vectors, of particular relevance to the present problem, are: (a) A toroidal vector, $A_g = \nabla \times (\hat{\boldsymbol{r}}\varrho\Psi_g)$, where, $\hat{\boldsymbol{r}}$ is the unit vector along $\boldsymbol{r}$, ϱ is introduced for later convenience, and Ψ_g is an arbitrary function of the coordinates. However, the boundary condition on Ψ_g, as imposed by Eq. (9 b), is $\Psi_g \to x$ as $x \to 0$ at the surface. (b) A poloidal vector, $A_t = \nabla \times \nabla \times (\hat{\boldsymbol{r}}\Psi_t)$, where the arbitrary function Ψ_t should remain finite at the surface to insure finiteness of the displacements.

The displacement vector generated by A_g is the poloidal vector $\zeta_g = \varrho^{-1}\nabla \times \nabla \times (\hat{\boldsymbol{r}}\varrho\Psi_g)$. This will later be identified as the generator of the g-components of the displacements and the subscript g is in anticipation of this property. The displacement generated by A_t is the torodial vector $\zeta_t = \varrho^{-1}\nabla \times \nabla \times \nabla \times (\hat{\boldsymbol{r}}\Psi_t)$. This will be identified with the neutral toroidal displacements of the fluid. Either by writing out ζ_g and ζ_t in their spherical polar coordinates or by some vector-algebraic manipulations, it can be easily verified that ζ_g and ζ_t are orthogonal in the sense of Eq. (2b). This completes the demonstration of the division of H_A into two orthogonal subspaces $H_g\{\zeta_g\}$ and $H_t\{\zeta_t\}$.

III. A Classification of the Modes of Self-gravitating Fluids

Let ϱ, p, and Ω be the density, the pressure, and the gravitational field of the fluid, respectively. The adiabatic Lagrangian displacements of the fluid, $\xi(\boldsymbol{r}) \exp(i\omega t)$, satisfy the following equation

$$\omega^2\varrho\xi = \nabla(\delta p) - \frac{1}{\varrho}\delta\varrho\nabla p - \varrho\nabla(\delta\Omega), \quad (12)$$

where

$$\delta\varrho = -\nabla \cdot (\varrho\xi), \quad (13a)$$

$$\delta p = \frac{dp}{d\varrho}\delta\varrho - \left[\left(\frac{\partial p}{\partial\varrho}\right)_{ad} - \frac{dp}{d\varrho}\right]\varrho\nabla \cdot \xi, \quad (13b)$$

$$\nabla^2\delta\Omega = 4\pi G\nabla \cdot (\varrho\xi). \quad (13c)$$

The adiabatic derivative of p with respect to ϱ is denoted by $(\partial p/\partial\varrho)_{ad}$ and the derivative prevailing in the equilibrium state is denoted by $dp/d\varrho$. From Eq. (12) one obtains the following variational expression for ω^2

$$\omega^2 = \frac{W}{S}, \quad (14)$$

where

$$S = \int \varrho\xi^* \cdot \xi dv, \quad (14a)$$

$$W = \int \frac{1}{\varrho}\frac{dp}{d\varrho}\delta^*\varrho\delta\varrho dv + \int \left[\left(\frac{\partial p}{\partial\varrho}\right)_{ad} - \frac{dp}{d\varrho}\right]\varrho\nabla \cdot \xi^*\nabla \cdot \xi dv - G\iint \delta^*\varrho(\boldsymbol{r})\delta\varrho(\boldsymbol{r}')|\boldsymbol{r}-\boldsymbol{r}'|^{-1} dv dv'. \quad (14b)$$

The variational Eq. (14) and its limited use in actual variational calculations can be found in Chandrasekhar (1964). The symmetry of the W-integral, however, was first demonstrated by Ledoux and Walraven (1958).

In classifying the modes the following theorem will be used: If two vectors ξ_1 and ξ_2 exist, such that $S(\xi_1+\xi_2) = S(\xi_1) + S(\xi_2)$ with no coupling term of the form $S(\xi_1, \xi_2)$, and similarly for W, then the variational Eq. (14) splits into two independent equations, $\omega_1^2 S(\xi_1) = W(\xi_1)$, and $\omega_2^2 S(\xi_2) = W(\xi_2)$. There is no coupling between Φ, A_g, and A_t terms in the S-integral. See Eq. (11) and the paragraphs following it. The corresponding density change, $\delta\varrho = \nabla \cdot (\varrho\nabla\Phi)$, is independent from A_g and A_t. Furthermore $\nabla \cdot \xi_t = 0$ on account of the fact that ϱ depends on $\boldsymbol{r}$ only. Thus, there is no coupling between Φ and A_t or between A_g and A_t in the W-integral. There is a coupling between Φ and A_g coming from the term containing $\nabla\cdot\xi$ in the W-integral. This, however, vanishes for a convectively neutral fluid, $(\partial p/\partial\varrho)_{ad}=dp/d\varrho$. Thus, by the theorem quoted above one concludes the following:

(i) There is a class of the neutral $[W(\xi_t)=0]$ modes of the fluid belonging to H_t subspace. This, of course, is a common knowledge.

(ii) There is a class of the normal modes of the convectively neutral modes which belong to H_Φ subspace. These are non-neutral $[W(\xi_\Phi)\neq 0]$ and have to be identified with the non-neutral p-modes of the fluid.

(iii) There is a class of the normal modes of the neutral fluid which belong to H_g subspace. These are neutral and have to be identified with the neutral degenerate g-, the only remaining modes of the fluid. Variational calculations of Sobouti (1976) and Sobouti and Silverman (1978), which employ ζ_Φ and ζ_g of this paper, confirm these identifications, numerically.

In a convectively non-neutral fluid the g-modes are non-neutral and non-degenerate. For small values of $(\partial p/\partial\varrho)_{ad}-(dp/d\varrho)$, however, they remain uncoupled from the p-modes. The W-integral for such g-modes is simply the integral containing $\nabla\cdot\xi$ in Eq. (14b). Thus,

$$W(\xi_g)=\int\left[\left(\frac{\partial p}{\partial\varrho}\right)_{ad}-\frac{dp}{d\varrho}\right]\frac{1}{\varrho^3}\left(\frac{d\varrho}{dr}\right)^2(\nabla\times A_g^*)_r(\nabla\times A_g)_r dv, \tag{15}$$

where $(\nabla\times A)_r$ is the radial component of $\nabla\times A$. Equation (15) in a different notation and with a different derivation is given by Smeyers (1966). An alternative derivation of it based on a perturbation technique and more details and numerical results are given by Sobouti and Silverman (1978).

Many of the physical characteristics of the g- and p-modes can be most easily deduced from the mathematical properties disclosed in this section. For example, the original observation of Cowling (1941), that the density and pressure variations are less pronounced in g-modes than in p-modes, becomes simply an expression of the fact that $\nabla\cdot(\nabla\times A)=0$, and the vector potential does not contribute to the Eulerian changes in ϱ. Or the observation that the self-gravitation term is less important in g-modes than in p-modes is similarly explained.

IV. An Expression for the Scalar Potential

Regarding the scalar potential Φ' there is a confusion in the literature. To clarify the situation we derive an explicit expression for Φ'. A formal solution of Eq. (4a) is

$$\Phi'(r)=\frac{1}{4\pi}\int_v \nabla'\cdot(\varrho\xi)|r-r'|^{-1}dv'$$

$$+\frac{1}{4\pi}\int_S\{\nabla'\Phi'|r-r'|^{-1}-\Phi'\nabla'(|r-r'|^{-1})\}\cdot dS'. \tag{16}$$

The first term in the surface integral vanishes by the boundary condition of Eq. (5a). On the other hand the solution of the Poisson Eq. (13c) for $\delta\Omega$ is

$$-\frac{1}{4\pi G}\delta\Omega=\frac{1}{4\pi}\int_v \nabla'\cdot(\varrho\xi)|r-r'|^{-1}dv'$$

$$+\frac{1}{16\pi^2 G}\int_S\{\nabla'\delta\Omega|r-r'|^{-1}-\delta\Omega\nabla'(|r-r'|^{-1})\}\cdot dS'. \tag{17}$$

This equation should be solved subject to the continuity of $\delta\Omega$ and $n\cdot\nabla\delta\Omega$ on the surface, which actually leads to vanishing of the surface integral. The two Eqs. (16) and (17) have the same source function, the volume integral. But the surface integrals are different, for the boundary conditions on Φ' and $\delta\Omega$ are not the same. Both inside and outside a surface, however, the potential due to a surface source distribution is a Laplace potential. Thus, we conclude that

$$\Phi'(r)=-\frac{1}{4\pi G}(\delta\Omega+\psi), \tag{18}$$

where

$$\nabla^2\psi=0. \tag{19}$$

The boundary value of ψ can be obtained as follows. From Eqs. (3) and (18)

$$\varrho\xi_r=\frac{1}{4\pi G}\left(\frac{\partial\delta\Omega}{\partial r}+\frac{\partial\psi}{\partial r}\right)+(\nabla\times A)_r. \tag{20}$$

At the surface R, the left hand side vanishes because of ϱ and $(\nabla\times A)_r$ vanishes because of Eq. (5b). Thus, one obtains

$$\left.\frac{\partial\psi}{\partial r}\right|_R=-\left.\frac{\partial\delta\Omega}{\partial r}\right|_R=(l+1)\frac{\delta\Omega(R)}{R}, \tag{21}$$

where the second equality is the boundary condition on the l-th. harmonic component of $\delta\Omega$ (Ledoux and Walraven 1958, pp. 513–514). With the boundary value of Eq. (21) the unique finite solution of the Laplace Eq. (19) is

$$\psi(r)=\frac{l+1}{l}C\left(\frac{r}{R}\right)^l Y_l^m(\theta,\phi),\ r<R \tag{22a}$$

$$=-C\left(\frac{R}{r}\right)^{l+1}Y_l^m(\theta,\phi), r>R, \tag{22b}$$

where $C=\delta\Omega(R)$ and an explicit expression for it is given below Eq. (23).

For the sake of completeness let us also give an expression for $\delta\Omega(r)$ (Cowling, 1941). The surface integral in Eq. (17) is zero. Expanding $|r-r'|^{-1}$ in Legendre polynominals and integrating over the r'-directions gives

$$\delta\Omega(r)=G\{r^{-(l+1)}\int_0^r\delta\varrho(r')r'^{l+2}dr'$$

$$+r^l\int_r^R\delta\varrho(r')r'^{-(l-1)}dr'\}Y_l^m(\theta,\phi),\ r<R, \tag{23a}$$

$$=C\left(\frac{R}{r}\right)^{l+1}Y_l^m(\theta,\phi),\ r>R, \tag{23b}$$

where $\delta\varrho(r)\,Y_l^m$ is the l-th. harmonic of $\delta\varrho(r)$ and $C=\delta\Omega(R)=G\int_0^R\delta\varrho(r')r'^{l+2}dr'/R^{l+1}$. From Eqs. (18), (22), and (23) one may now construct the potential Φ'. As a demonstration of consistency we observe that: a) Outside the star both Φ' and $\nabla\Phi'$ are identically zero. b) On the surface of the star $\varrho\xi\to-\nabla\Phi'\to 0$. The potential Φ' itself, however, is discontinuous across the surface. This is due to the existence of the surface integral in Eq. (16). Aizenman and Smeyers (1977) do not make the distinction between the different boundary conditions for Φ' and $\delta\Omega$. As a result they identify Φ' with $\delta\Omega$, missing the ψ-term (see their Eq. 13 and the comments thereof).

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