Astron. Astrophys. 321, 1024–1026 (1997)

ASTRONOMY AND ASTROPHYSICS

Response of a star to gravitational waves

H.G. Khosroshahi[1] **and Y. Sobouti**[1,2,3]

[1] Institute for Advanced Studies in Basic Sciences, P.O. Box 45195-159, Zanjan, Iran (iasbsgz1@rose.ipm.ac.ir, sobouti@rose.ipm.ac.ir)
[2] Department of physics, Shiraz University, Shiraz, Iran
[3] Center for Theoretical Physics and Mathematics, AEOI, Tehran, Iran

Received 12 June 1996 / Accepted 21 October 1996

Abstract. We investigate the possibility of the excitation of the oscillation modes of polytropic stars by gravitational waves. We decompose the displacement vector field of a normal mode into its irrotational and solenoidal components and show that the interaction with the gravitational waves takes place through the irrotational component. We calculate the absorption cross section of the waves for different modes and find that the cross section for p modes are orders of magnitudes larger than the g modes. In the p sequence the cross section is largest for the fundamental mode and decreases with increasing mode order.

Key words: gravitational waves – stars: oscillation – methods: numerical

1. Introduction

Weber (1968) suggested to consider the Earth as a gravitational wave detector in the millihertz band. Since then many authors have elaborated on both theoretical and observational aspects of the problem(Dyson 1969, Ashby and Dreitlein 1975). In quest for still lower frequencies in the microhertz regions, Mashhoon and his collaborators studied the secular changes in the orbital parameters of binaries and in the Earth-Moon and Earth-Mars distances (Mashhoon 1979, Mashhoon et al. 1981, Anderson & Mashhoon 1985).

In this paper we study excitation of the normal modes of a star by gravitational waves. The behavior of the *g* and *p* modes of oscillation and details of their coupling to gravitational waves are analyzed. The cross section for resonant absorption of the wave energy are calculated. Numerical values are given for several modes of polytropic models.

2. Normal modes of a star

The formalism below is parallel to that of Beiki and Sobouti (1990) who studied excitation of the oscillation of a binary member by its companion. Consider a non rotating spherical star in hydrostatic equilibrium. Let a mass element at $\mathbf{r}$ adiabaticly undergo an infinitesimal lagrangian displacement $\xi(\mathbf{r}, t)$ from its equilibrium position. Let $\delta\rho$, δp and $\delta\Omega$ denote the corresponding Eulerian changes in the density, ρ, the pressure, p and the gravitational potential, Ω, respectively. The linearized Euler's equation of motion is

$$-\rho\ddot{\xi} = \nabla(\delta p) + \delta\rho\nabla\Omega + \rho\nabla(\delta\Omega) = \mathscr{W}\xi \ , \tag{1}$$

where

$$\delta\rho = -\nabla.(\rho\xi) = -\nabla\rho.\xi - \rho\nabla.\xi \ , \tag{2}$$

$$\delta p = \frac{dp}{d\rho}\delta\rho - [(\frac{\partial p}{\partial \rho})_{ad} - \frac{dp}{d\rho}]\rho\nabla.\xi. \tag{3}$$

$$\nabla^2(\delta\Omega) = -4\pi G\delta\rho. \tag{4}$$

All terms in Eq. (1) are expressed in terms of the vector field ξ. The second equality in this equation is the definition for the operator $\mathscr{W}$ whose properties will be discussed shortly.

3. The Hilbert space of the displacement field

Let $\mathscr{H}$ be a function space whose elements are $\xi(\mathbf{r})$ and in which the inner product is defined as

$$(\eta, \rho\xi) = \int \eta^*.\rho\xi \ d^3x = finite, \quad \xi, \eta \in \mathscr{H}, \tag{5}$$

where the integration is over the volume of the star. $\mathscr{W}$ on $\mathscr{H}$ is self adjoint, $(\eta, \mathscr{W}\xi) = (\mathscr{W}\eta, \xi)$. There follows the eigenvalue problem

$$\mathscr{W}\xi_n = \omega_n^2\rho\xi_n, \tag{6}$$

where ω_n is the eigenfrequency of an oscillation mode, ξ_n is its eigendisplacement vector, and *n* is a collection of three indices, indicating the three wave numbers in, say, $(\hat{\mathbf{r}}, \hat{\theta}, \hat{\phi})$ directions of a spherical polar coordinates. Furthermore, $\{\xi_n\}$ is an orthogonal set and can be normalized to unity,

$$(\xi_n, \rho\xi_m) = \delta_{nm}. \tag{7}$$

The set $\{\xi_n\}$ is also complete and may serve as a basis for $\mathscr{H}$. See Dixit et al. (1980). Thus, any $\xi(\mathbf{r}, t)$ may be expanded in a unique way in terms of $\{\xi_n\}$. Numerical values of $\{\omega_n, \xi_n\}$ for hypothetical or actual star models are abundant in the astronomical literature of seventies and eighties.

Send offprint requests to: H.G. Khosroshahi

4. The g and p components of the displacement field

Using a gauged version of Helmholtz theorem (Sobouti 1981), one may decompose a general displacement vector into an irrotational and a "weighted" solenoidal component. Thus

$$\xi = \xi_p + \xi_g, \tag{8}$$

where

$$\xi_p = -\nabla\chi_p; \qquad \text{with} \nabla\times\xi_p = 0, \tag{8a}$$
$$\xi_g = \rho^{-1}\nabla\times A = \rho^{-1}\nabla\times\nabla\times(\hat{\mathbf{r}}\chi_g); \text{with} \nabla.(\rho\xi_g) = 0. \tag{8b}$$

Here $\hat{\mathbf{r}}$ is the unit vector in r direction, and $\chi_p(\mathbf{r})$ and $\chi_g(\mathbf{r})$ are two scalars. Evidently both components are poloidal and mutually orthogonal, $(\xi_p, \rho\xi_g)$=0. Next, we define the dimensionless parameter $\epsilon = (\rho/p)[dp/d\rho - (\partial p/\partial\rho)_{ad}]$. Schwarzschild's criterion for convective neutrality is $\epsilon = 0$. In such a fluid one readily sees that $\mathscr{W}\xi_p = \omega_p^2\rho\xi_p$ and $\mathscr{W}\xi_g = 0$. That is, the oscillatory motions of the fluid are of purely p-type and are driven mainly by the compression forces $-\nabla\delta p$. The g-type motions are neutral, $\omega_g^2 = 0$. It can also be shown (Sobouti & Silverman 1978) that to the first order of smallness in ϵ, the p oscillations retain their pure p nature of Eq. (8a), while the g motions develop into a sequence of new and long period oscillations of the type of Eq. (8b). The latter are driven by the buoyancy forces, $-\delta\rho\nabla\Omega$. For larger values of ϵ the two types get more and more coupled. One last remark: If $\epsilon > 0$, then $\omega_g^2 > 0$; the fluid is stable to convective motions and the oscillatory g modes develop. If $\epsilon < 0$ then $\omega_g^2 < 0$ and convective motions are set up. The imaginary frequency, ω_g, then indicates the rate of the exponential growth of the convective motions.

Note added in revision: Toroidal modes of the fluid in the present scheme have the form $\xi_t = \nabla\times\nabla\times\nabla(\hat{\mathbf{r}}\chi_t)$, $\chi_t(\mathbf{r})$ =scalar. In the Newtonian models considered in this paper they are neutral and remain neutral upon exposure to gravitational waves. The referee, however, has informed us of a recent work of Kokkotas (in press) where he shows that in relativistic stars they also are excited by gravitational waves.

5. Interaction with gravitational waves

The perturbation of the metric tensor associated with a weak gravitational wave propagating in the z-direction may be written as,

$$h_{\mu\nu}(\mathbf{x}, t) = \Re\{A_{\mu\nu}e^{i(kz-\omega t)}\}, \tag{9}$$

where $\omega = ck$, and $A_{\mu\nu}$, in a transverse-traceless gauge, is

$$A_{xx} = -A_{yy} = A_+, A_{xy} = A_{yx} = A_\times, \text{all others zero.} \tag{9a}$$

Here A_+ and $A_\times$ are the amplitudes of the two orthogonal polarizations of the wave. We shall assume $A_\times = 0$ and the wavelength much longer than dimensions of the star (i.e. $e^{ikz} \approx 1$). The relevant non vanishing components of the Riemann curvature tensor are then

$$R^1{}_{010} = -R^2{}_{020} = -\frac{1}{2}h_{11,00} = \frac{1}{2}\omega^2 A_+(\omega)e^{-i\omega t}. \tag{10}$$

The geodesics of a mass element at $\mathbf{x}$ will deviate from that of the center of mass at the origin upon exposure to the gravitational wave. The corresponding acceleration is

$$\frac{d^2x^i}{dt^2} = -R^i{}_{0j0}x^j. \tag{11}$$

By Eq. (10), however, this can be written as

$$\ddot{\mathbf{x}} = -\frac{1}{2}\omega^2 A_+(\omega)e^{-i\omega t}\nabla V, \tag{12}$$

where the potential V is

$$V = \frac{1}{2}(x^2 - y^2) = \sqrt{\frac{2\pi}{15}}r^2[Y_{2,2}(\theta,\phi) + Y_{2,-2}(\theta,\phi)]. \tag{12a}$$

Coming back to the oscillations of the star, we add the Newtonian force term associated with the acceleration of Eq. (12) to the right hand side of Eq. (1). To account for attenuation of the motion within the star we also postulate a friction force proportional to the velocity, $2\gamma\rho\dot{\xi}$, γ=const. Thus,

$$\rho(\ddot{\xi} + 2\gamma\dot{\xi}) + \mathscr{W}\xi = -\frac{1}{2}\omega^2\rho A_+(\omega)e^{-i\omega t}\nabla V. \tag{13}$$

This is the equation of a damped wave driven by the external force of the gravitational radiation. We solve it by expanding $\xi(\mathbf{r}, t)$ in terms of the normal modes (ω_n, ξ_n) of the $\mathscr{W}$ operator, Eq. (6). Thus

$$\xi(\mathbf{r}, t) = \sum_m c_m\xi_m(\mathbf{r})e^{-i\omega t}. \tag{14}$$

Substituting Eq. (14) in Eq. (13), multiplying the resulting expression by $\xi_n^*(\mathbf{r})$ and integrating over the volume of the star gives

$$c_n = \frac{1}{2}A_+(\omega)\frac{\omega^2}{(\omega^2 - \omega_n^2) + 2i\gamma\omega}(\xi_n, \rho\nabla V). \tag{14a}$$

5.1. Energy absorption

Time dependence of $\xi(\mathbf{r}, t)$ and of the gravitational force is periodic. The time-averaged rate of the energy transfer to the star becomes

$$\frac{d\bar{E}(\omega)}{dt} = \frac{1}{2}Re\int\dot{\xi}^*(\mathbf{r}, t)\cdot(-\frac{1}{2}\omega^2\rho A_+(\omega)e^{-i\omega t}\nabla V)d^3x. \tag{15}$$

Substituting for $\dot{\xi}$ from Eq. (14) and simple manipulations gives

$$\frac{d\bar{E}(\omega)}{dt} = \frac{1}{4}\sum_n\frac{\gamma A_+^2(\omega)\omega^6}{(\omega^2 - \omega_n^2)^2 + 4\gamma^2\omega^2}\mid(\xi_n, \rho\nabla V)\mid^2. \tag{15a}$$

We assume that the amplitude $A_+(\omega)$ remains reasonably constant over the Lorenzian profile. One can then see that the resonances occur at $\omega_n + O(\gamma^2)$ with half-width $2\gamma + O(\gamma^3)$ and maxima $(\omega_n^4/4\gamma)(1 + O(\gamma^2))$. Thus, the total energy absorption, proportional to the area $\omega_n^4/2$ of the line profiles, becomes

$$(\frac{d\bar{E}}{dt})_{tot} = \frac{1}{8}\sum_n A_+^2(\omega_n)\omega_n^4|(\xi_n, \rho\nabla V)|^2. \tag{16}$$

Table 1. Cross sections for different modes of polytropic indeces 1.5, 2, 2.5

	Polytropic Index 1.5			Polytropic Index 2			Polytropic Index 2.5		
mode	ω_n^2	$\|(\xi_n, \rho \bigtriangledown V)\|^2$	σ_n	ω_n^2	$\|(\xi_n, \rho \bigtriangledown V)\|^2$	σ_n	ω_n^2	$\|(\xi_n, \rho \bigtriangledown V)\|^2$	σ_n
p_5	6.345(+1)	6.3867(-7)	6.365(-5)	6.112(+1)	1.5946(-5)	1.531(-3)	5.979(+1)	1.4613(-4)	1.372(-2)
p_4	4.130(+1)	1.0238(-5)	6.641(-4)	4.063(+1)	1.0914(-4)	6.965(-3)	4.068(+1)	5.8340(-4)	3.727(-2)
p_3	2.351(+1)	1.8897(-4)	6.978(-3)	2.407(+1)	8.7635(-4)	3.313(-2)	2.512(+1)	2.6647(-3)	1.051(-1)
p_2	1.029(+1)	5.2094(-3)	8.420(-2)	1.155(+1)	9.6116(-3)	1.743(-1)	1.314(+1)	1.5576(-2)	3.214(-1)
p_1	2.119(0)	4.0377(-1)	1.343(0)	3.113(0)	2.9824(-1)	1.458(0)	4.832(0)	1.9896(-1)	1.510(0)
g_1	0	0	0	5.633(-1)	7.4242(-4)	6.569(-4)	1.805(0)	4.3563(-3)	1.235(-2)
g_2	0	0	0	2.967(-1)	1.2377(-4)	5.768(-5)	9.904(-1)	8.9175(-4)	1.387(-3)
g_3	0	0	0	1.839(-1)	2.4745(-5)	7.148(-6)	6.284(-1)	2.3766(-4)	2.345(-4)
g_4	0	0	0	1.254(-1)	6.0549(-6)	1.192(-6)	4.192(-1)	8.0304(-5)	5.287(-5)
g_5	0	0	0	8.858(-2)	1.1506(-6)	1.600(-7)	2.646(-1)	2.0701(-5)	8.604(-6)

Frequencies, ω_n^2, are in units of $3.95 \times 10^{-7}(M_*/M_\odot)(R_\odot/R_*)^3\ sec^{-2}$.
Cross sections, σ_n, are in units of $G^2\rho_c M_* {R_*}^2/\gamma c^3$.
For $\rho_{c_\odot} = 16\ gr/cm^3$, $R_\odot = 6.96 \times 10^{10}\ cm,\ M_\odot = 2 \times 10^{33}\ gr$ and $\gamma = 1\ s^{-1}$, this unit is $2.5 \times 10^{10}\ cm^2$.

5.2. Absorption cross section

The energy flux of the gravitational waves per unit frequency interval is

$$\Phi(\omega) = \frac{c^3}{16\pi G} < \dot{h}_{xx}^2 + \dot{h}_{yy}^2 >_{time\ av.} = \frac{c^3 A_+^2(\omega)\omega^2}{8\pi G}. \quad (17)$$

The cross section for the energy transfer from the gravitational waves to the star is the ratio of $(d\bar{E}/dt)$ to $2\gamma\Phi(\omega_n)$, the total flux across the resonant profile. Thus

$$\sigma_{tot} = \frac{\pi}{2}\frac{G}{\gamma c^3}\sum_n \omega_n^2 |(\xi_n, \rho\nabla V)|^2. \quad (18)$$

6. The overlap integrals

Using the g and p decomposition of Eq. (8) for ξ_n gives

$$\begin{aligned}(\xi_n, \rho\nabla V) = & \int \rho(-\nabla\chi_n^* + \tfrac{1}{\rho}\nabla \times A_n^*).\nabla V d^3x \\ = & -\int \rho\nabla\chi_n^*.\nabla V d^3x = \int \chi_n^*\nabla\rho.\nabla V d^3x \end{aligned} \quad (19)$$

By Eq. (12), V is a spherical harmonic of order 2. Therefore, only the normal modes belonging to $l = 2,\ m = \pm 2$ will contribute to the overlap integral, that is, $\chi_n(\mathbf{r}) = \chi_n(r)Y_{2,\pm 2}(\theta, \phi)$. Substituting for χ_n and V and carrying out integrations over θ, ϕ gives

$$(\xi_n, \rho\nabla V) = 4\sqrt{\frac{2\pi}{15}}\int \frac{d\rho}{dr}\chi_n(r)r^3 dr. \quad (19a)$$

For numerical calculations the following steps were taken.

1) A Rayleigh-Ritz variational method was employed to obtain the eigenfrequencies and eigenfunctions for various g and p modes(Sobouti & Silverman 1978). The method consisted of expanding the p and g potentials of Eqs. (8) in power series of r, substituting the resulting ξ's in Eq. (6) and finding the expansion coefficients by variational calculations.

2) The information thus obtained was used to extract the χ_p potential for each of the p and g modes and to calculate the overlap integrals of Eq. (19a), and eventually the cross sections and the energy absorption rates. Numerical values for polytropic structures are summarized in Table 1.

7. Concluding remarks

The gravitational radiation, being a quadrupole one and derivable from a scalar potential excites only the second order harmonic modes of the star and that only through the irrotational component, χ_n. Therefore, the g-modes with small irrotational components present much smaller absorption cross sections to the gravitational radiation than the p modes. In the p sequence the cross section decreases as the mode order goes up. It is largest for p_1, commonly known as the fundamental mode in Cowling's (1941) terminology. See Table 1 for these behaviors. The analysis following Eq. (9) was carried out for a (+) polarized radiation. The result for a $(\times)$ polarization and, consequently, for unpolarized radiation will, however, be the same.

Acknowledgements. We would like to thank Professor B. Mashhoon for helpful discussions and suggestions.

References

Anderson J., Mashhoon B., 1985, ApJ 290, 445
Ashby N., Dreitlen J., 1975, Phys. Rev. D12, 336
Beiki Ardakani A., Sobouti Y., 1990, A&A 227, 71
Cowling T.G., 1941, MNRAS 101,367
Dixit V.V., Sarath B., Sobouti Y., 1980, A&A 89, 259
Dyson F.J., 1969, ApJ 156, 529
Kokkotas K.D., "Pulsating Relativistic Stars", Proc. of the Les Houches Summer School, Eds. J.A. Mark and J.P. Lasota, Cambridge Univ. press, in press (gr-qc/9603024)
Mashhoon B., 1979, ApJ 227, 1019
Mashhoon B., Carr B.J., Hu B.L., 1981, ApJ 246, 569
Sobouti Y., Silverman, J.N., 1978, A&A 62, 365
Sobouti Y., 1981, A&A 100, 319
Weber J., 1968, Phys. Rev. Lett. 21, 395

This article was processed by the author using Springer-Verlag LaTeX A&A style file *L-AA* version 3.