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Convective Modes and Convective Stability of Rotating Fluids*

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Summary. Axisymmetric modes of a rotating self-gravitating fluid are analyzed. In the limit of small deviations from convective neutrality and slow rotation, the following conclusions have been reached. The g-modes of a rotating fluid are affected only by the Coriolis forces. In a superadiabatic fluid, the g-modes with large radial wave numbers and small non-radial wave numbers are suppressed. The extent of suppression depends rather critically on the ratio of *a measure of rotation* to a *measure of superadiabaticity* of the fluid. The g-modes with sufficiently small radial wave numbers and sufficiently large non-radial wave numbers, however, remain unstable. The criterion for stability of a rotating fluid remains the same as the Schwarzschild criterion.

Key words: convection in rotating fluids — normal modes: self gravitating fluids — stability of rotating fluids

I. Introduction

In a superadiabatic fluid, rotation inhibits part of the unstable convective motions. Evidently, Randers' displacements (1942), primarily in radial directions, and Walen's displacements (1946), primarily perpendicular to the axis of rotation, fall in this category. Cowling (1951), however, argued that rotation cannot have an absolute stabilizing effect on convection. Under certain simplifying assumptions he showed that displacements of sufficiently large horizontal wave number remain unstable.

Since Cowling's work certain developments have enhanced our understanding of the problem. Chandrasekhar and Lebovitz (1968) have shown that the axisymmetric displacements of a rotating system are solutions of an ordinary eigenvalue problem. Eisenfeld (1969) has demonstrated the completeness of the normal modes of non-rotating fluids. Sobouti (1977a) has clarified the g- and p-classification of the modes. Sobouti and Silverman (1977) have shown that the g- and p-modes of a convectively neutral fluid are solutions of two independent eigenvalue problems. Finally the last authors have removed the degeneracy of the neutral convective spectrum. In the light of these findings, a more systematic analysis of Cowling's problem under less stringent assumptions seems possible.

This paper is concerned with linear axisymmetric convection in rotating inviscid fluids. The problem is treated as one of normal modes analysis. Equations of motion and expansion of modes in terms of a basis set, consisting of g-, p- and *toroidal*-components, are introduced in Section II. This is done on the basis of Sobouti's definition of the g- and p-modes (1977a, hereafter referred to as paper I). A double perturbation procedure, in which the operators pertaining to a non-adiabatic rotating system are expanded about those of an adiabatic non-rotating fluid, is discussed in Section III. This is done as an extension of Silverman and Sobouti's perturbational-variational scheme (1978, hereafter referred to as Paper II). Partitioning of modes and separation of their g-component (the genesis of convective motions) is carried out in Section IV. This is done according to Sobouti and Silverman's separation scheme (1978, hereafter referred to as Paper III). Stability criterion is discussed in Section V. Cutoff wave numbers, separating the surviving convective modes from disrupted ones (by rotation) are obtained in Section VI. The various matrices appearing in the formalism are calculated in Appendix A.

II. Equations of Motion

Let, p, ρ and U denote the pressure, the density, and the gravitational potential of the fluid, respectively, and let Ω be its constant rotation vector about the z-axis. The equilibrium state of the system is given by

$$\nabla p - \rho\nabla[U + \tfrac{1}{2}\Omega^2(x^2 + y^2)] = 0. \quad (1)$$

For future reference, let us recall that, as rotation tends to zero, the solutions ρ, p, and U of Equation (1) tend

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continuously to the corresponding solutions of the non-rotating system. A perturbation expansion of these quantities in power series of Ω^2 is possible.

Time-separated adiabatic Lagrangian displacements of the fluid, $\boldsymbol{\xi}^s(\mathbf{r}) \exp(i\sqrt{\varepsilon^s}t)$ (if they exist), are governed by the following equations:

$$\mathscr{W}\boldsymbol{\xi}^s + 2i\sqrt{\varepsilon^s}\rho\boldsymbol{\Omega} \times \boldsymbol{\xi}^s - \varepsilon^s\rho\boldsymbol{\xi}^s = 0, \tag{2}$$

where

$$\mathscr{W}\boldsymbol{\xi} = \nabla(\delta p) - \frac{1}{\rho}\nabla p\delta\rho - \rho\nabla(\delta U), \tag{2a}$$

$$\delta p = -\gamma p\nabla\cdot\boldsymbol{\xi} - \nabla p\cdot\boldsymbol{\xi}, \tag{2b}$$

$$\delta\rho = -\rho\nabla\cdot\boldsymbol{\xi} - \nabla\rho\cdot\boldsymbol{\xi}, \tag{2c}$$

$$\nabla^2(\delta U) = -4\pi G\delta\rho. \tag{2d}$$

In the absence of rotation, Equation (2) constitutes an eigenvalue problem. The corresponding eigensolutions have a simple discrete spectrum and form a complete orthonormal set in a Hilbert space $\mathscr{H}_0$. The inner product in $\mathscr{H}_0$ is defined as $\int \rho_0\boldsymbol{\zeta}^{r*}\cdot\boldsymbol{\zeta}^s dv$, where ρ_0 is the density of the non-rotating system. With rotation present, Equation (2) is no longer an eigenvalue equation in the conventional sense of the word. It can be shown, however, that solutions $\boldsymbol{\xi}^s$, of Equation (2) belong to a vector space $\mathscr{H}$ in which the inner product is $\int \rho\boldsymbol{\xi}^{r*}\cdot\boldsymbol{\xi}^s dv$. One observes that both ρ and ρ_0 are bounded and have bounded domains. Also ρ has a convergent expansion about ρ_0. From these observations, one concludes that solutions $\boldsymbol{\xi}^s$ of Equation (2) belong to the Hilbert space $\mathscr{H}_0$, the span of the normal modes of the non-rotating system. Therefore, one will be allowed to expand $\boldsymbol{\xi}^s$ in terms of the complete eigensolutions of the non-rotating fluid or in terms of any other basis set for that space.

Thus, let $\{\boldsymbol{\zeta}^r; r = 1, 2, \ldots\}$ be a suitable basis set for $\mathscr{H}_0$ and let the displacement $\boldsymbol{\xi}^s$ of Equation (2) be expanded in terms of this set:

$$\boldsymbol{\xi}^s = \sum_r \boldsymbol{\zeta}^r Z^{rs}. \tag{3}$$

A matrix formalism will be followed throughout the paper. Let Z be the matrix of the expansion coefficients above and E and $E^{1/2}$ be two diagonal matrices whose elements are ε^s and $\sqrt{\varepsilon^s}$; $s = 1, 2, \ldots$, respectively. Thus,

$$Z = [Z^{rs}]; r, s = 1, 2, \ldots, \tag{3a}$$

$$E = \begin{vmatrix} \varepsilon^1 & & \\ & \varepsilon^2 & \\ & & \ddots \end{vmatrix}, \tag{3b}$$

$$E^{1/2} = \begin{vmatrix} \sqrt{\varepsilon^1} & & \\ & \sqrt{\varepsilon^2} & \\ & & \ddots \end{vmatrix}. \tag{3c}$$

Also, let the matrices representing the various operators in Equation (2) be denoted as follows:

$$W^{sr} = W^{rs*} = \int \boldsymbol{\zeta}^{s*}\cdot\mathscr{W}\boldsymbol{\zeta}^r dv, \tag{4a}$$

$$C^{sr} = -C^{rs*} = \frac{1}{\Omega}\int \rho\boldsymbol{\zeta}^{s*}\cdot(\boldsymbol{\Omega} \times \boldsymbol{\zeta}^r)dv$$

$$= -\frac{1}{\Omega}\int \rho\boldsymbol{\Omega}\cdot(\boldsymbol{\zeta}^{s*} \times \boldsymbol{\zeta}^r)dv, \tag{4b}$$

$$S^{sr} = S^{rs*} = \int \rho\boldsymbol{\zeta}^{s*}\cdot\boldsymbol{\zeta}^r dv. \tag{4c}$$

Hermitian character of the $\mathscr{W}$-operator and, therefore, that of the W-matrix is established by Clement (1964) and by Lynden-Bell and Ostriker (1967). The former author generalizes a variational principle of Chandrasekhar (1964), governing linear adiabatic oscillations of self-gravitating fluids, to a uniformly rotating system. The latter authors extend the principle further, to fluids possessing steady internal motions in an inertial or in a rotating reference system. Anti-Hermitian nature of the Coriolis matrix C, and Hermitian and positive definite character of the S-matrix are evident from their defining Equations (4b) and (4c), respectively.

In terms of the matrices of Equations (3) and (4), the matrix representation of Equation (2) becomes

$$WZ + 2i\Omega CZE^{1/2} - SZE = 0. \tag{5}$$

To obtain the (rs) element of Equation (5), one substitutes the expansion of Equation (3) for $\boldsymbol{\xi}^s$ in Equation (2), premultiplies by $\boldsymbol{\zeta}^{r*}$ and integrates over the volume of the fluid.

The normal modes of the non-rotating system are of three types: the g-modes, the p-modes and the *toroidal*-modes, of which the latter are neutral. Accordingly the basis set $\{\boldsymbol{\zeta}^r\}$ employed in this paper consists of the g- and p-subsets, $\{\boldsymbol{\zeta}_g^r\}$ and $\{\boldsymbol{\zeta}_p^r\}$, of Paper I, and a third toroidal-subset $\{\boldsymbol{\zeta}_t^r\}$. A spherical harmonic expansion of the basis vectors $\boldsymbol{\zeta}^r$, and of the Lagrangian displacement vectors $\boldsymbol{\xi}^s$ of equations (2) and (3) will be considered below. Because of the axial symmetry of the equilibrium configuration, however, the Lagrangian displacements belonging to a given azimuthal harmonic number m, will not be coupled to others and are dealt with independently. In the remainder of this paper only axisymmetric displacements, belonging to $m = 0$, will be considered. The spherical harmonic expansions of these axisymmetric basis vectors are

$$\boldsymbol{\zeta}_\varepsilon^{rl}: \left(\frac{1}{r^2}\psi_\varepsilon^{rl}Y_l, \frac{1}{l(l+1)}\frac{1}{r}\chi_\varepsilon^{rl'}\frac{\partial Y_l}{\partial\theta}, 0\right); \varepsilon = g, p, \tag{6a}$$

$$\boldsymbol{\zeta}_t^{rl}: \left(0, 0, -\frac{\psi_t^{rl}}{r^2}\frac{\partial Y_l}{\partial\theta}\right), \tag{6b}$$

where each vector is now specified by a pair of superscripts (rl), of which the first index indicates the radial mode order and the second index denotes the horizontal mode order of the vectors. The indices r and l will be referred to as the radial wave number and the horizontal or non-radial wave number of the vector in question, respectively. The scalar functions ψ and χ for the g- and p-basis vectors are related as follows (see Paper I, Section II):

$$\chi_g^{rl'} = \psi_g^{rl'} + \frac{p_0'}{\gamma p_0}\psi_g^{rl}, \tag{7a}$$

$$\chi_p^{rl''} = l(l+1)\frac{1}{r^2}\psi_p^{rl}, \tag{7b}$$

where p_0 is the pressure of the non-rotating and convectively neutral fluid, henceforth used as the reference system. By virtue of Equations (7), we observe that each of the g-, p-, and *toroidal*-basis vectors depend on only one single scalar function, say ψ_g, ψ_p, and ψ_t, respectively.

Let the set of basis vectors $\{\boldsymbol{\zeta}\}$, and the set of the eigen-displacements $\{\boldsymbol{\xi}\}$ be partitioned into their non-toroidal and toroidal components as follows:

$$\{\boldsymbol{\zeta}\} = \{\boldsymbol{\zeta}_\varepsilon | \boldsymbol{\zeta}_t\}, \tag{8a}$$

$$\{\boldsymbol{\xi}\} = \{\boldsymbol{\xi}_\varepsilon | \boldsymbol{\xi}_t\}, \tag{8b}$$

where the subscript ε stands for the g- and p-components of the set, combined together. Equation (3), written out explicitly, assumes the following form

$$\boldsymbol{\xi}_\varepsilon^s = \sum_r \boldsymbol{\zeta}_\varepsilon^r Z_{\varepsilon\varepsilon}^{rs} + \sum_q \boldsymbol{\zeta}_t^q Z_{t\varepsilon}^{qs}, \tag{8c}$$

$$\boldsymbol{\zeta}_t^s = \sum_r \boldsymbol{\zeta}_\varepsilon^r Z_{\varepsilon t}^{rs} + \sum_q \boldsymbol{\zeta}_t^q Z_{tt}^{qs}. \tag{8d}$$

Equations (8) entail a corresponding partitioning of all the matrices entering Equation (5). Thus,

$$A = \left[\begin{array}{c|c} A_{\varepsilon\varepsilon} & A_{\varepsilon t} \\ \hline A_{t\varepsilon} & A_{tt} \end{array}\right];\ A = W, C, S, Z, E^{1/2}, E. \tag{9}$$

Elementary arguments will reveal simpler structures of the W-, C- and S-matrices. The Eulerian variations of the pressure, the density, and the gravitational potential generated by a toroidal vector, are identically zero. Therefore, the W-matrix assumes the following form:

$$W = \left[\begin{array}{c|c} W_{\varepsilon\varepsilon} & 0 \\ \hline 0 & 0 \end{array}\right]. \tag{10a}$$

A simple geometrical reasoning shows that the diagonal $\varepsilon\varepsilon$- and tt-blocks of the Coriolis matrix C vanish. Let us consider the triple product $\boldsymbol{\Omega}\cdot(\boldsymbol{\zeta}^{s*} \times \boldsymbol{\zeta}^r)$ in the defining Equation (4b). If both basis vectors are of g- and/or p-type, then the three vectors $\boldsymbol{\Omega}$, $\boldsymbol{\zeta}^{s*}$ and $\boldsymbol{\zeta}^r$ lie in the same meridian plane. The triple product and consequently the $C_{\varepsilon\varepsilon}^{sr}$ elements vanish. If both $\boldsymbol{\zeta}^{s*}$ and $\boldsymbol{\zeta}^r$ are of toroidal type, then both are in the azimuthal direction. Their vector product and consequently the C_{tt}^{sr} elements vanish. The C-matrix thus becomes:

$$C = \left[\begin{array}{c|c} 0 & C_{\varepsilon t} \\ \hline C_{t\varepsilon} & 0 \end{array}\right]. \tag{10b}$$

The toroidal vectors of Equation (6b) are perpendicular to the non-toroidal vectors of Equation (6a). This makes the scalar product $\boldsymbol{\zeta}_\varepsilon^*\cdot\boldsymbol{\zeta}_t$ and consequently the non-diagonal blocks of the S-matrix vanish. Thus,

$$S = \left[\begin{array}{c|c} S_{\varepsilon\varepsilon} & 0 \\ \hline 0 & S_{tt} \end{array}\right]. \tag{10c}$$

Substitution of Equations (10a)–(10c) in Equation (5) and block-multiplication of the various terms gives

$$\left[\begin{array}{c|c} W_{\varepsilon\varepsilon}Z_{\varepsilon\varepsilon} + 2i\Omega C_{\varepsilon t}Z_{t\varepsilon}E_\varepsilon^{1/2} - S_{\varepsilon\varepsilon}Z_{\varepsilon\varepsilon}E_\varepsilon & W_{\varepsilon\varepsilon}Z_{\varepsilon t} + 2i\Omega C_{\varepsilon t}Z_{tt}E_t^{1/2} - S_{\varepsilon\varepsilon}Z_{\varepsilon t}E_t \\ \hline 2i\Omega C_{t\varepsilon}Z_{\varepsilon\varepsilon}E_\varepsilon^{1/2} - S_{tt}Z_{t\varepsilon}E_\varepsilon & 2i\Omega C_{t\varepsilon}Z_{\varepsilon t}E_t^{1/2} - S_{tt}Z_{tt}E_t \end{array}\right] = 0. \tag{11}$$

This equation will be studied blockwise:

(i) The tt-block of Equation (11), in view of the fact that S_{tt} is positive definite and Z_{tt} cannot be zero, gives

$$E_t = E_t^{1/2} = 0. \qquad Z_{tt} \text{ remains undetermined.} \tag{12a}$$

(ii) The εt-block, complemented with Equation (12a), gives

$$Z_{\varepsilon t} = 0. \tag{12b}$$

(iii) The $t\varepsilon$-block can be considered as a relation between $Z_{t\varepsilon}$ and $Z_{\varepsilon\varepsilon}$. Provided that E_ε is non-singular, one may solve this relation for $Z_{t\varepsilon}$. Thus,

$$Z_{t\varepsilon} = 2i\Omega S_{tt}^{-1}C_{t\varepsilon}Z_{\varepsilon\varepsilon}E_\varepsilon^{1/2}. \tag{13}$$

If E_ε contains a zero eigenvalue of multiplicity m, then the m columns of Equation (11), belonging to this vanishing eigenvalue, remain undetermined. Equation (13) and the subsequent developments based on this equation still remain valid, provided that the zero elements of E_ε and the corresponding m columns and/or rows of $Z_{\varepsilon\varepsilon}$ and $C_{t\varepsilon}$ are eliminated from these matrices.

(iv) Substitution of Equation (13) in the $\varepsilon\varepsilon$-block of Equation (11) gives

$$(W_{\varepsilon\varepsilon} - 4\Omega^2 C_{\varepsilon t}S_{tt}^{-1}C_{t\varepsilon})Z_{\varepsilon\varepsilon} - S_{\varepsilon\varepsilon}Z_{\varepsilon\varepsilon}E_\varepsilon = 0. \tag{14}$$

In Appendix B we use the closure property of the toroidal basis set $\{\boldsymbol{\zeta}_t\}$ to show that the Coriolis term $4\Omega^2 C_{\varepsilon t}S_{tt}^{-1}C_{t\varepsilon}$ is independent of this $\{\boldsymbol{\zeta}_t\}$ set. The elements of the Coriolis term are completely specified by a pair of poloidal vectors from the $\{\zeta_\varepsilon\}$ set. Thus, Equation (14) becomes:

$$(W_{\varepsilon\varepsilon} + 4\Omega^2 R_{\varepsilon\varepsilon})Z_{\varepsilon\varepsilon} - S_{\varepsilon\varepsilon}Z_{\varepsilon\varepsilon}E_\varepsilon = 0, \tag{14a}$$

where $R_{\varepsilon\varepsilon}$ denotes the Coriolis term $C_{\varepsilon t}S_{tt}^{-1}C_{t\varepsilon}$. Its elements are given by Equation (B7) of Appendix B:

$$\Omega^2 R_{\varepsilon\varepsilon}^{pq} = \Omega^2 R_{\varepsilon\varepsilon}^{qp*} = -\Omega^2(C_{\varepsilon t}S_{tt}^{-1}C_{t\varepsilon})^{pq} = = \int \rho(\boldsymbol{\Omega} \times \boldsymbol{\zeta}_\varepsilon^{p*})\cdot(\boldsymbol{\Omega} \times \boldsymbol{\zeta}_\varepsilon^{q})dv. \tag{B7}$$

The matrix $W_{\varepsilon\varepsilon} + 4\Omega^2 R_{\varepsilon\varepsilon}$ in Equation (B7) is Hermitian. The matrix $S_{\varepsilon\varepsilon}$ is Hermitian and positive definite. Therefore, Equation (14a) is an eigenvalue problem for $Z_{\varepsilon\varepsilon}$ and E_ε. The eigenvalues are real and are the roots of the secular determinant:

$$|W_{\varepsilon\varepsilon} + 4\Omega^2 R_{\varepsilon\varepsilon} - \varepsilon S_{\varepsilon\varepsilon}| = 0. \tag{14b}$$

The eigenvectors, that is, the columns of the $Z_{\varepsilon\varepsilon}$-matrix, form an orthonormal set in the following sense.

$$Z_{\varepsilon\varepsilon}^\dagger S_{\varepsilon\varepsilon} Z_{\varepsilon\varepsilon} = \mathrm{I}. \tag{14c}$$

An immediate conclusion from the real-valuedness of the eigenvalues (the squares of the eigenfrequencies) is that the stability or instability of inviscid rotating fluids is of dynamical nature. No secular instability, associated with axisymmetric displacements, develops on account of rotation of the fluid.

In Appendix B we also reduce Equation (13) and show that the toroidal component of Equation (8c) is completely determined by the poloidal component of this same vector. Thus Equations (8c), (13) and (B10) give

$$\boldsymbol{\xi}_\varepsilon^s = \left(1 + \frac{2i}{\sqrt{\varepsilon^s}}\boldsymbol{\Omega} \times\right)\boldsymbol{\xi}_{\varepsilon\varepsilon}^s, \tag{15}$$

where $\boldsymbol{\xi}_{\varepsilon\varepsilon}$ is the poloidal component of $\boldsymbol{\xi}_\varepsilon$ and is given by

$$\boldsymbol{\xi}_{\varepsilon\varepsilon}^s = \sum_\tau \boldsymbol{\zeta}_\varepsilon Z_{\varepsilon\varepsilon}^{\tau s}. \tag{15a}$$

To summarize the conclusions of this section we construct the E- and Z-matrices from Equations (12)–(14):

$$E = \left|\begin{array}{c|c} E_\varepsilon & 0 \\ \hline 0 & 0 \end{array}\right|; \qquad E_\varepsilon \text{ diagonal}, \tag{16a}$$

$$Z = \left|\begin{array}{c|c} Z_{\varepsilon\varepsilon} & 0 \\ \hline Z_{t\varepsilon} & Z_{tt} \end{array}\right|, \tag{16b}$$

where E_ε and $Z_{\varepsilon\varepsilon}$ are solutions of Equations (14a)–(14c) and Z_{tt} remains indeterminate.

Equations (14a) and (15) are respectively equivalent to Equations (24) and (16) of Chandrasekhar and Lebovitz (1968). In the present approach, however, the problem of the normal modes of a rotating system is looked into as one in the theory of linear vector spaces. The expansion in terms of basis sets is emphasized. The full spectrum of the modes, its possible partitioning into g-, p- and *toroidal*-components, and their subsequent interactions are studied systematically. The main reason for the present derivation, however, has been to arrive at the general Equation (5). For non-axisymmetric displacements, the simplifications of Equations (10)–(15) do not hold. The toroidal modes are likely to be excited by the g- and p-modes and develop into a non-neutral non-degenerate spectrum. Equation (5) and the expansions of Equations (8) will then serve as a logical starting point.

III. A Procedure for the Analysis of Equations (14)

It was shown in Paper III that, in the limit of small deviations from convective neutrality, the eigenvalue equation for the non-rotating fluid can be partitioned into two independent equations, one for each of the g- and p-modes. This was accomplished by considering a perturbation expansion of the normal modes of the non-adiabatic system about those of the adiabatic one. In rotating fluids a similar separation of modes is possible and considerably simplifies the problem.

The reference fluid about which the perturbation expansion is carried out is the non-rotating and convectively neutral fluid. The quantities pertaining to this reference system will be denoted by a subscript zero. One of the perturbation parameters will indicate deviation of the actual fluid from the adiabatic limit and will be chosen as follows:

$$a = \frac{\partial \ln p}{\partial \ln \rho} \Big/ \frac{\partial \ln p_0}{\partial \ln \rho_0} - 1. \tag{17a}$$

We shall only be concerned with systems for which a is constant throughout the fluid. For example, this is the case for a polytrope of index n and of constant ratio of specific heats γ:

$$a = \frac{1}{\gamma}\left(1 + \frac{1}{n}\right) - 1. \tag{17b}$$

The second perturbation parameter will be a measure of the uniform rotation of the fluid:

$$b = 4\Omega^2 \frac{(\gamma - 1)}{4\pi G \rho_{c0}\gamma}, \tag{17c}$$

where ρ_{c0} is the central density of the reference fluid. In this description, the pressure and the density of the fluid at any point $\mathbf{r}$ will further depend on a and b. For example, $p = p(\mathbf{r}, a, b)$. Next, the structural parameters of the fluid will be Taylor-expanded in terms of a and b. Up to the first order terms one obtains

$$\sigma(\mathbf{r}, a, b) = \sigma_0(\mathbf{r}) + a\sigma_a(\mathbf{r}) + b\sigma_b(\mathbf{r}); \qquad \sigma = p, \rho, \tag{18a}$$

where

$$\sigma_a(\mathbf{r}) = \left[\frac{\partial}{\partial a}\sigma(\mathbf{r}, a, 0)\right]_{a=0}, \sigma_b(\mathbf{r}) = \left[\frac{\partial}{\partial b}\sigma(\mathbf{r}, 0, b)\right]_{b=0}. \tag{18b}$$

Note that $\mathbf{r}$, a and b are treated as independent variables. The operators ∇, $\partial/\partial a$ and $\partial/\partial b$ commute with each other.

The expansion of Equations (18) is different from that of Simon (1969). In the present scheme one expands the pressure and density of the actual fluid at point r about the pressure and density of the reference system at the same point r. Simon expands the pressure and density of an element of the actual fluid about the pressure and density of a corresponding element in the reference fluid. The distinction between the present expansion and that of Simon is analogous to the distinction between the Eulerian time derivative $\partial/\partial t$ and the Lagrangian time derivative d/dt in fluid dynamics.

The double expansion of Equations (18), via the defining Equations (4) and (B7), results in a corresponding expansion of the W-, R-, and S-matrices:

$$A(a, b) = A_0 + aA_a + bA_b; \qquad A = W_{\varepsilon\varepsilon}, R_{\varepsilon\varepsilon}, S_{\varepsilon\varepsilon}. \quad (19)$$

A double perturbation expansion of the W-, R- and S-matrices can at most imply similar expansions for the p-modes. The non-existence of the perturbation expansion for the g-modes can be seen in two ways. (1) The neutral g-state of the unperturbed fluid is infinitely degenerate. Each of the perturbing forces arising from non-adiabaticity and from rotation of the fluid are capable of removing the degeneracy. But each force generates its own set of orthogonal normal modes which are incompatible with those of the other force. See for example Hirschfelder et al. (1964) for an exposition of this issue. (2) Let the non-adiabaticity parameter a be fixed and consider rotation as the only perturbing force. The g-eigenvalues of the non-rotating and non-adiabatic fluid, which now serves as the reference system, all tend to zero with increasing radial wave numbers. Among the g-modes of the reference system, with assorted radial and non-radial wave numbers, one can find an infinite number of pairs of eigenvalues which are arbitrarily close to each other. Since the inverse of the difference of such pairs of eigenvalues enter the perturbation series, convergence of the expansion series is not guaranteed. With these remarks in mind let us write

$$Z_{\varepsilon\varepsilon} = Z_{0\varepsilon\varepsilon} + Z_{1\varepsilon\varepsilon} \quad (20a)$$

and

$$E_\varepsilon = E_{0\varepsilon} + E_{1\varepsilon}, \quad (20b)$$

where $Z_{0\varepsilon\varepsilon}$ and $E_{0\varepsilon}$ are the eigensolutions of the reference fluid:

$$W_{0\varepsilon\varepsilon}Z_{0\varepsilon\varepsilon} - S_{0\varepsilon\varepsilon}Z_{0\varepsilon\varepsilon}E_{0\varepsilon} = 0. \quad (21)$$

It is shown in Paper III and it will be recapitulated in Section IV that Equation (21) is an eigenvalue equation for the p-modes alone. Equation (21) does not contain information on the g-modes and accordingly does not impose restrictions on the g-components of $Z_{0\varepsilon\varepsilon}$ and $E_{0\varepsilon\varepsilon}$. We are thus at liberty to consider $Z_{1\varepsilon\varepsilon}$ and $E_{1\varepsilon\varepsilon}$ in Equations (20) as first order quantities. With regards to their p-components, this follows from the existence of perturbation series. With regards to their g-components, this is an option. The option exists because no restrictions on the g-components of $Z_{0\varepsilon\varepsilon}$ and $E_{0\varepsilon\varepsilon}$ are as yet imposed. Note that no attempt is made, and with regard to the g-modes it is not allowed, to decompose $Z_{1\varepsilon\varepsilon}$ and $E_{1\varepsilon\varepsilon}$ into terms proportional to a and b.

Next we substitute Equations (19) and (20) in Equation (14a) and separate the zeroth and the first order terms. In the zeroth order one recovers Equation (21). In the first order one obtains

$$(aW_a + bW_b + bR_0)Z_0 + W_0Z_1 - S_0Z_0E_1 - S_0Z_1E_0 - (aS_a + bS_b)Z_0E_0 = 0. \quad (22)$$

For brevity, the subscripts ε on all the matrices of Equation (22) are suppressed. An analysis of Equations (21) and (22) is carried out in Section IV.

IV. Partitioning of Equations (21) and (22) According to the g- and p-Classification of the Modes

We recall that the basis set $\{\zeta_\varepsilon\}$ responsible for generation of the matrix Equations (14a), (21) and (22) consists of two g- and p-subsets. The g-subset $\{\zeta_g\}$ is given by Equations (6a) and (7a), and spans the g-subspace of the normal modes of the reference fluid. The p-subset $\{\zeta_p\}$ is given by Equation (6a) and (7b), and spans the p-subspace of the normal modes of the reference fluid. This partitioning of the basis set results in a corresponding partitioning of the vectors and matrices appearing in Equations (14)–(22). For the matrices one obtains

$$A = \left[\begin{array}{c|c} A_{gg} & A_{gp} \\ \hline A_{pg} & A_{pp} \end{array}\right]; \qquad A = W_{\varepsilon\varepsilon}, R_{\varepsilon\varepsilon}, \text{etc.} \quad (23)$$

(i) *The Zeroth Order Solutions*

A review of Paper III, Section IV. From Equations (13) of paper III one has

$$W_{0\varepsilon\varepsilon} = \left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & W_{0pp} \end{array}\right], \quad (24a)$$

$$S_{0\varepsilon\varepsilon} = \left[\begin{array}{c|c} S_{0gg} & 0 \\ \hline 0 & S_{0pp} \end{array}\right]. \quad (24b)$$

This form of the W- and S-matrices in turn leads to the following solutions for E_0 and Z_0 [cf. Eqs. (18) of Paper III]:

$$E_{0\varepsilon} = \left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & E_{0p} \end{array}\right], \qquad E_{0p} \text{ diagonal}, \quad (25a)$$

$$Z_{0\varepsilon\varepsilon} = \left[\begin{array}{c|c} Z_{0gg} & 0 \\ \hline 0 & Z_{0pp} \end{array}\right], \quad (25b)$$

where Z_{0gg} remains indeterminate in the zeroth order, and Z_{0pp} and E_{0p} are solutions of

$$W_{0pp}Z_{0pp} = S_{0pp}Z_{0pp}E_{0p}. \tag{26}$$

(ii) The first Order Solutions

Solutions of Equation (22), among other information, provide the first order corrections to the p-eigenvalues arising from non-adiabaticity and rotation of the fluid. By the assumption of small deviation from convective neutrality and slow rotation, however, these are first order corrections aE_{ap} and bE_{bp} to the non-vanishing E_{0p}. They will not in any case change the positive sign of the p-eigenvalues. Therefore, none of the p-modes can be expected to become unstable. Perturbations of the p-modes will not be pursued in this paper. The situation for the g-modes, however, is different. The zeroth order g-eigenvalues E_{0g} are all zero [cf. Equations (25a)]. The first order quantities E_{1g} are the first nonvanishing terms in Equation (20b). Their negative or positive signs will decide whether or not convection should take place. The matrix E_{1g} along with Z_{0gg} are solutions of the gg-block of Equation (22). This block will be analyzed in detail.

The matrices $W_{a\varepsilon\varepsilon}$, $R_{0\varepsilon\varepsilon}$, $S_{a\varepsilon\varepsilon}$, and $S_{b\varepsilon\varepsilon}$ have no vanishing blocks. We shall see in Appendix A, Equation (A13), that $W_{b\varepsilon\varepsilon}$ has the following simplified structure:

$$W_{b\varepsilon\varepsilon} = \left[\begin{array}{c|c} 0 & W_{bgp} \\ \hline W_{bpg} & W_{bpp} \end{array}\right]. \tag{27}$$

Substituting Equations (24), (25) and (27) in Equation (22) (note the subscript ε in all the matrices of this last equation was suppressed) and separating the gg-block of the resulting equation gives

$$(aW_{agg} + bR_{0gg})Z_{0gg} = S_{0gg}Z_{0gg}E_{1g}. \tag{28}$$

Equation (28) is a generalization of Equation (21a) of Paper III to include the effects of rotation, the b-term. This is an eigenvalue equation for Z_{0gg} with E_{1g} as the corresponding eigenvalue matrix. Equation (28) will be analyzed to the extent of finding answers to two questions: (a) what is the criterion for convective stability of a rotating fluid? and (b) what are the convective modes which survive the stabilizing influence of rotation? To achieve this goal explicit expressions for the elements of W_{agg} and R_{0gg} are needed. These quantities are discussed in Appendix A. The questions (a) and (b) above, are pursued in Sections V and VI, respectively.

V. Stability Criterion

The question of stability of the fluid is narrowed down to the question of positive-definiteness of the total matrix $aW_{agg} + bR_{0gg}$ of Equation (28). From Equations (A14) and (A15) we see that $-W_{agg}$ is positive definite and R_{0gg} is non-negative. Therefore, for a subadiabatic fluid ($a < 0$), the total matrix will be positive definite and the fluid will be stable for g- or convective motions. Ordinarily, completeness of the eigensolutions and of the basis vectors is required to ascertain that all possible displacements of the fluid are exhausted as linear combination of the eigendisplacements, and to conclude stability. We note, however, that the positive definiteness of the matrices in question is inferred from positive definiteness of their generating operators. Lebovitz (1966) employs a theorem by Laval et al. (1965) to show that the Schwarzschild criterion can be obtained from the positive definite nature of the operator alone. One can in fact use his reasoning in the present problem. In doing so, one will not have to appeal to basis vectors or to the eigendisplacements. The question of completeness will not arise.

For a superadiabatic fluid ($a > 0$), the total matrix $aW_{agg} + bR_{0gg}$ can never be positive definite. We demonstrate this by showing that, for any value of a (> 0) and b, some of the diagonal elements of the total matrix are negative. (We recall that a matrix is positive definite if and only if all of its principal minors are positive. The diagonal elements of a matrix are among its principal minors. They all should be positive for positive definiteness.) The expression $p_0'\rho_0'/\rho_0$, in the integrand of Equation (A14b), decreases to zero at the surface more slowly than ρ_0 in the integrand of Equation (A19). This can easily be verified by expressing p_0 and ρ_0 in terms of the polytropic variable θ, and recalling that at the surface of polytrope, θ is zero and its derivative is finite. Apart from the two factors mentioned, the remainder of the integrands in Equations (A14b) and (A19) are the same. On the other hand, the function ψ^{sk} in the integrands contain the power r^k [see Eq. (A18)]. Consequently, the larger the k grows, the more the surface layers contribute to the integrals. Because of these two factors [i.e. (a) the slower decrease of the integrand of $W_{0gg}^{sk,sk}$ than the integrand of $R_{0gg}^{sk,sk}$ and (b) the larger contribution of the surface layers at large horizontal wave numbers k] one concludes the following:

For any finite values of a (>0), b and s one can always find a number K such that, if $k > K$, the expression $aW_{agg}^{sk,sk} + R_{0gg}^{sk,sk} < 0$.

This completes the demonstration of the non-positive nature of the total matrix, $aW_{agg} + bR_{0gg}$. The essence of the argument has been to show that, at large horizontal wave numbers, the destabilizing matrix aW_{agg} is more dominant than the stabilizing Coriolis term bR_{0gg}. The opposite is true for large radial wave numbers. Because of the factor ts in the asymptotic Equation (A20), the Coriolis term becomes dominant at large radial wave numbers s. Completeness of the eigenvectors of Equation (28) is not required to draw the last conclusion. The argument merely is that, among the convective displacements of Equations (6a) and (7a) one finds unstable disturbances which cannot be suppressed by the Coriolis forces. At this stage, again one can abandon the matrix formalism and arrive at the stability criterion by looking into the nature of the operator giving rise to Equation (28) or, equivalently, (14a). This operator is $\mathscr{W} - 4\rho\boldsymbol{\Omega} \times \boldsymbol{\Omega} \times$. For a partially or totally superadiabatic fluid, this is not positive definite: The scalar product $\int \zeta_g^{sk*} \cdot [\mathscr{W} - 4\rho\boldsymbol{\Omega} \times \boldsymbol{\Omega} \times]\zeta_g^{sk} dv$ corresponding to large k's is negative. The proof is already given; the prescribed integral is no more than a diagonal element of the total matrix $aW_{agg} + bR_{0gg}$.

Figures 1 and 2 are intended to serve as an alternative numerical and graphical verification of the statements of this section. For various values of s and k, the diagonal elements, $W_{agg}^{sk,sk}$ and $R_{0gg}^{sk,sk}$ have been computed from the original Equations (A14b) and (A17b), respectively. In Figure 1, these elements are plotted as function of k while s is kept constant. The slopes of the W-curves are less steep than the slopes of the corresponding R-curves. Therefore, the destabilizing W-term dominates at large horizontal wave numbers. In Figure 2, the elements are plotted as functions of s while k is kept constant. Here, the slopes of the R-curves are gentler than the slopes of the W-curves, indicating that the stabilizing Coriolis forces dominate at large radial wave numbers. Let us conclude this section by recapitulating that:

Rotation does not, in the mathematical sense of the word, stabilize a totally or partially superadiabatic fluid. The criterion for absolute stability of the fluid remains the same as in the absence of rotation; that is, the Schwarzschild criterion.

The conclusion is the same as that of Cowling. The mode of argument, however, is substantially different and the assumptions are considerably less stringent.

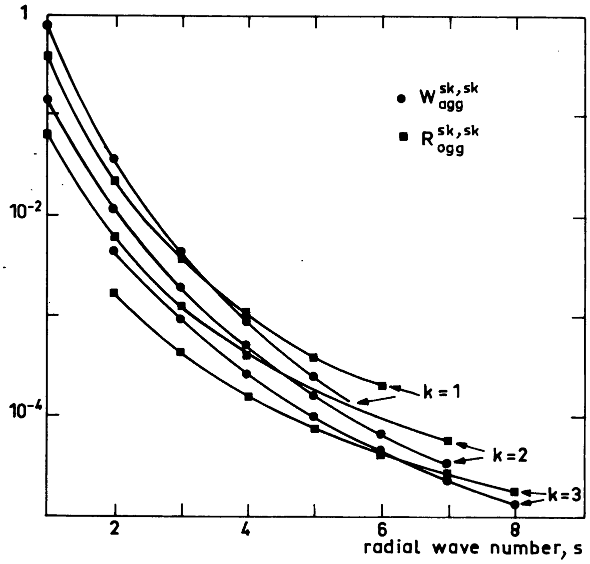

Fig. 1. Diagonal elements of W_{agg} and R_{0gg} are plotted as functions of the radial wave number. At large radial wave numbers, W_{agg} decreases more rapidly than R_{0gg}

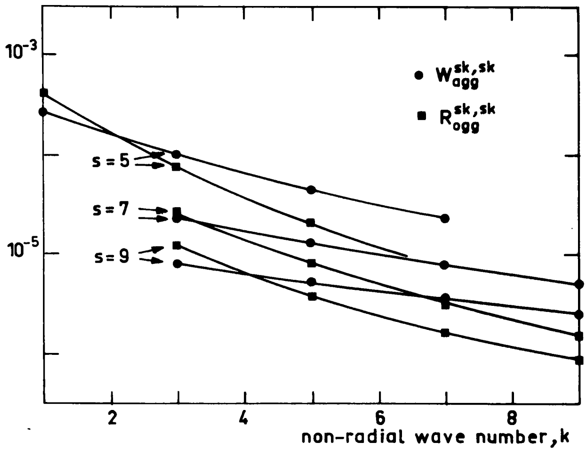

Fig. 2. Diagonal elements of W_{agg} and R_{0gg} are plotted as functions of the non-radial wave number. At large non-radial wave numbers, R_{0gg} decreases more rapidly than W_{agg}

VI. Cutoff Wave Numbers

Consider a plane of wave numbers (s, k), Figure 3. Each eigensolution ξ_g^{sk} of Equation (28) is represented by a point in this plane. From Figures 1 and 2 one learns that, in the region of large s and small k, Coriolis forces dominate the forces arising from superadiabaticity, and vice versa. Thus, in a superadiabatic fluid convective motions of small radial extent and of large horizontal dimensions are disrupted by the Coriolis forces and are replaced by stable displacements. These stable models will have no resemblance to convective motions. Remaining unstable motions will, more or less, keep their convective nature. Corresponding eigenvalues (square of time rates of exponential growth), however, will diminish.

For a given ratio of b/a [divide Equation (28) by a to see the emergence of this ratio] and a given number s_c on the s-axis one can always find a corresponding k_c on the k-axis such that the modes ξ_g^{sk}, with $s < s_c$ and $k > k_c$ are of unstable type, while those corresponding to $s > s_c$ and $k < k_c$ have become stabilized. Let us call the pair (s_c, k_c) the cutoff wave numbers. The collection of these cutoff number pairs will divide the (s, k) plane into two regions of stable and unstable modes. To obtain the pair of cutoff numbers one should solve Equation (28) for a given value of b/a and then sort out the combinations of smallest k and largest s for which the eigenvalues are negative. In the present work, however, the cutoff pairs are obtained by a much simpler but approximate method. For a given ratio b/a and a given s_c, the diagonal elements $[W_{agg} + (b/a)R_{0gg}]^{s_c k, s_c k}$ are computed for an increasing sequence of k's until the sum becomes negative. The value of k in the sequence, for which the diagonal element just becomes negative, is approximately k_c. The argument in support of this procedure is the following: suppose the set of the actual eigendisplacements $\{\xi_g^{sk}\}$ is

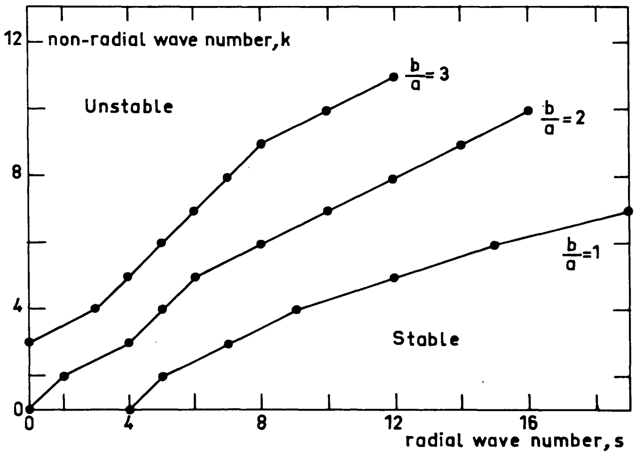

Fig. 3. In the wave number plane (s, k), each mode is represented by a point (with integer coordinates). The collection of cutoff numbers (s_c, k_c), for a given ratio b/a, divide the (s, k) plane into two regions. The points to the left of the cutoff lines are unstable convective modes. The points to the right of and on the cutoff lines represent the stabilized modes. As the ratio b/a increases, the cutoff line moves to the left and more connective modes are suppressed

used as the basis set. The matrix $W_{agg} + (b/a)R_{0gg}$ would then be the diagonal matrix E_1 and the process just described to obtain (s_c, k_c) would be exact. Instead, the $\{\zeta_g^{sk}\}$ of Equations (6a), (7a), and (A18) is used as the basis set. This set, however, has proved to be a good approximation to eigendisplacements of the reference system $\{\xi_{0g}\}$ (see Paper I and Sobouti, 1977b). The latter set, in turn, is the asymptotic limit of the convective branch of $\{\xi_g^{sk}\}$.

Figure 3 shows sample plots of (s_c, k_s) for three values of b/a. The region to the left of each curve is the unstable zone. The modes falling in this zone are of convective type and, evidently, are burdened with the task of heat transfer in the star. In agreement with Tayler (1973) we see that the extent of suppression of convective modes grows as the ratio b/a increases.

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Appendix A: The W- and R-Matrices

Only the gg-block of W, R, and their derivatives with respect to a and b are needed in Equation (28). To calculate these, one in turn requires perturbation expansion of the Eulerian changes in pressure and density. From proposition I of Paper I, we recall that ζ_g's are those displacements of the reference fluid which leave the pressure equilibrium undisturbed. That is

$$\delta_g p_0 = -\gamma p_0 \nabla\cdot\zeta_g - \nabla p_0\cdot\zeta_g = 0 \tag{A1a}$$

(for the time being the superscripts on the basis vectors ζ_g are suppressed). Via Equation (A1a), we further find that the Eulerian variations of the density and of the gravitational potential of the reference fluid also vanish (see Paper III, Appendix A):

$$\delta_g\rho_0 = \rho_0\nabla\cdot\zeta_g - \nabla\rho_0\cdot\zeta_g = 0, \tag{A1b}$$

$$\delta_g U_0 = G\int |\mathbf{r}-\mathbf{r}'|^{-1}\delta_g\rho_0(\mathbf{r}')dv' = 0. \tag{A1c}$$

The first order a and b terms of Eulerian variations are:

$$\delta_g p_c = -\gamma p_c\nabla\cdot\zeta_g - \nabla p_c\cdot\zeta_g, \tag{A2a}$$

$$\delta_g\rho_c = -\rho_c\nabla\cdot\zeta_g - \nabla\rho_c\cdot\zeta_g, \tag{A2b}$$

$$\delta_g U_c = G\int |\mathbf{r}-\mathbf{r}'|^{-1}\delta_g\rho_c(\mathbf{r}')dv', \tag{A2c}$$

where the subscript c is either a or b. In both the actual and the reference fluid, the equilibrium pressure is a function of the density. This, in turn, results in corresponding relations between the first order Eulerian variations of p and ρ. These relations are obtained below.

(i) The Relation between $\delta_g p_b$ and $\delta_g\rho_b$

Let us write the pressure-density relation as follows:

$$p = p(\rho); \qquad \rho = \rho(\mathbf{r}, a, b). \tag{A3a}$$

Similarly, for the reference fluid

$$p_0 = K\rho_0^\gamma; \qquad \rho_0 = \rho_0(\mathbf{r}). \tag{A3b}$$

From Equations (18b) and (A3) one obtains

$$p_b = \left[\frac{\partial p}{\partial b}\right]_{a=b=0} = \left[\frac{dp}{d\rho}\frac{\partial\rho}{\partial b}\right]_{a=b=0} = \frac{\gamma p_0}{\rho_0}\rho_b. \tag{A4}$$

Substitution of Equation (A4) in Equation (A2a) for $\delta_g p_b$ and repeated use of Equation (A1a) to eliminate the term $\nabla p_0\cdot\zeta_g$, wherever occurring, gives

$$\delta_g p_b = \frac{\gamma p_0}{\rho_0}\delta_g\rho_b. \tag{A5}$$

(ii) The Relation between $\delta_g p_a$ and $\delta_g\rho_a$

To understand the distinction between the two cases for a and b, let us consider the example of polytropes:

$$p = K\rho^{[1+(1/n)]} = K\rho^{\gamma(a+1)}; \qquad \rho = \rho(\mathbf{r}, a, b), \tag{A6}$$

where we have used Equation (17b) to eliminate n in favor of a. Equation (A6) states that p, in addition to being function of ρ, is also an explicit function of a. Thus, a more general equation of state than Equation (A3a) is

$$p = p(a, \rho); \qquad \rho = \rho(\mathbf{r}, a, b). \tag{A7}$$

On differentiating Equation (A7) with respect to a, one obtains

$$p_a = \left[\left(\frac{\partial p}{\partial\rho}\right)_a\frac{\partial\rho}{\partial a} + \left(\frac{\partial p}{\partial a}\right)_\rho\right]_{a=b=0}. \tag{A8a}$$

Note the second term on the right hand side of Equation (A8a). There is no counterpart to this term in Equation (A4) for p_b. For polytropes, Equation (A8a) becomes

$$p_a = \frac{\gamma p_0}{\rho_0}\rho_a + \gamma p_0 \ln \rho_0. \tag{A8b}$$

Substitution of Equation (A8b) in Equation (A2a) gives

$$\delta_g p_a = \frac{\gamma p_0}{\rho_0}\delta_g \rho_a + \gamma p_0 \nabla\cdot\boldsymbol{\zeta}. \tag{A9}$$

Equation (A9), though derived for polytropes, is, however, general and holds for any fluid structure. A simple derivation of Equation (A9) in the general case is obtained by expressing $\delta_g p$ of Equation (A2) of Paper III in terms of $\delta_g \rho$ of Equation (A3) of the same paper and then differentiating the result with respect to a.

We are now ready to compute the matrix elements. The pair of superscripts which designated a basis vector will be restored; thus, $\boldsymbol{\zeta}_g^{tl}$ [cf. Eq. (6a)]. The Eulerian variation of Equations (A1) and (A2) will also be specified by the superscripts of their corresponding vector. For example, $\delta_g^{tl} p_a$ will denote the first order pressure change (with respect to a) induced by $\boldsymbol{\zeta}_g^{tl}$. Correspondingly, a typical matrix element $W_{gg}^{tl,sk}$ will be designated by two pairs of superscripts of its generating vectors $\boldsymbol{\zeta}_g^{tl}$ and $\boldsymbol{\zeta}_g^{sk}$.

a) The W-Matrix

After some integrations by parts, the defining Equations (4a) and (2a–d) give

$$W_{gg}^{tl,sk} = -\int \nabla\cdot\boldsymbol{\zeta}_g^{tl}\delta_g^{sk} p\, dv - \int \frac{1}{\rho}\boldsymbol{\zeta}_g^{tl}\cdot\nabla p\,\delta_g^{sk}\rho\, dv - \int \delta_g^{tl}\rho\,\delta_g^{sk} U dv. \tag{A10}$$

A perturbation expansion of Equation (A10) is obtained by substituting the expansions of Equations (18) for p and ρ and the expansions of Equations (A1) and (A2) for various Eulerian variations and separating the zeroth and the first order terms. The zeroth order matrix vanishes:

$$W_{0gg}^{tl,sk} = 0. \tag{A11}$$

The first order matrices become

$$W_{cgg}^{tl,sk} = \int \nabla\cdot\boldsymbol{\zeta}^{tl}\left(\gamma\frac{p_0}{\rho_0}\delta_g^{sk}\rho_c - \delta_g^{sk} p_c\right) dv, \tag{A12}$$

where $c = a$ or b. In deriving Equation (A12), Equation (A1) is used to eliminate $\boldsymbol{\zeta}_g^{tl}\cdot\nabla p_0$ from the second term in the integrand. The self-gravitation term, the third integral in Equation (A10), has no contribution to the first order matrices W_{cgg}. This is because of the product $\delta_g \rho\, \delta_g U$ in the third integral, which vanishes in both the zeroth and the first order. This provides a justification for the common belief that, at least in the course of convective processes, the effects of self-gravitation are negligible.

Substitution of Equation (A5) in Equation (A12) gives

$$W_{bgg}^{tl,sk} = 0. \tag{A13}$$

We recall that W_{bgg} arises from rotation because of the fact that the pressure and the density of the rotating fluid are different from those of the non-rotating one. That is, W_{bgg} is the contribution of the centrifugal forces. This, however, vanishes in the first order. Therefore, we conclude that the Coriolis forces alone affect the convective motions (a) by assigning a toroidal component to each convective motion [see Equation (15)], and (b) by altering the eigenfrequencies and eigendisplacements of the convective modes [see the R-matrix in Equation (28)].

Substitution of Equation (A9) in Equation (A12) gives

$$W_{agg}^{tl,sk} = -\int \gamma p_0 \nabla\cdot\boldsymbol{\zeta}_g^{tl}\nabla\cdot\boldsymbol{\zeta}_g^{sk} dv. \tag{A14a}$$

Employing the spherical harmonic expansion of $\boldsymbol{\zeta}_g$, Equation (6a), and using Equation (7a) to eliminate the non-radial component of $\boldsymbol{\zeta}_g$, and integrating over the solid angles, reduces Equation (A14a) to the following

$$W_{agg}^{tl,sk} = -\delta_{lk}\int \frac{p_0'\rho_0'}{\rho_0}\psi_g^{tl}\psi_g^{sk}\frac{dr}{r^2}. \tag{A14b}$$

The integral in Equation (A14a) becomes positive for any arbitrary $\boldsymbol{\xi}$ substituted simultaneously for $\boldsymbol{\zeta}_g^{tl}$ and $\boldsymbol{\zeta}_g^{sk}$. Therefore, $-W_{agg}$ is positive definite. The same can be inferred from Equation (A14b).

b) The R-Matrix

From the defining equation (B7) one has

$$R_{0gg}^{tl,sk} = \int \rho_0 \zeta_{g\varpi}^{tl}\,\zeta_{g\varpi}^{sk} dv, \tag{A15}$$

where $\zeta_{g\varpi}$ is the component of $\boldsymbol{\zeta}_g$ normal to the axis of rotation. From Equation (6a)

$$\zeta_{g\varpi}^{tl} = \frac{1}{r^2}\psi_g^{tl} Y_l \sin\theta + \frac{1}{l(l+1)}\frac{1}{r}\chi_g^{tl'}\frac{\partial Y_l}{\partial\theta}\cos\theta. \tag{A16}$$

We note that ρ_0, ψ's and χ's are functions of r only. From the symmetries of the spherical harmonic functions and the particular form of the angular integrals appearing in Equation (A15), it follows that all elements of the R-matrix, for which l is different from k or $k \pm 2$, vanish:

$$R_{0gg}^{tl,ks} = 0; \quad l \neq k \quad \text{or} \quad k \pm 2. \tag{A17a}$$

For the non-vanishing elements, on substituting Equation

(A16) in Equation (A15) and carrying out the angular integrations, one obtains

$$R_{0gg}^{tk,sk} = R_{0gg}^{sk,tk} = \frac{1}{(2k-1)(2k+3)}$$

$$\left\{2(k^2+k-1)\int \rho_0 \psi_g^{tk}\psi_g^{sk}\frac{dr}{r^2}\right.$$

$$-\int \rho_0[\psi_g^{tk}\chi_g^{sk'} + \psi_g^{sk}\chi_g^{tk'}]\frac{dr}{r}$$

$$\left.+\frac{(2k^2+2k-3)}{k(k+1)}\int \rho_0 \chi_g^{tk'}\chi_g^{sk'}dr\right\}, \tag{A17b}$$

$$R_{0gg}^{tk,sk+2} = R_{0gg}^{sk+2,tk} = \frac{[(2k+1)(2k+5)]^{1/2}}{(2k+1)(2k+3)(2k+5)}$$

$$\left\{-(k+1)(k+2)\int \rho_0\psi_g^{tk}\psi_g^{sk+2}\frac{dr}{r^2}\right.$$

$$+\int \rho_0[-(k+1)\psi_g^{tk}\chi_g^{sk+2'} + (k+2)\psi_g^{sk+2}\chi_g^{tk'}]\frac{dr}{r}$$

$$\left.+\int \rho_0\chi_g^{tk'}\chi_g^{sk+2'}dr\right\}. \tag{A17c}$$

The elements corresponding to $l = k - 2$ are obtained from Equation (A17c) by replacing k with $k - 2$.

Asymptotic Behavior of the Elements of R-matrix at Large Wave Numbers. The scalar functions, χ_g', appearing in Equations (A17), are given by Equation (7a):

$$\chi_g^{tl'} = \psi_g^{tl} + \frac{\rho_0'}{\rho_0}\psi_g^{tl}. \tag{7a}$$

A possible ansatz for ψ_g is suggested in Section V of Paper I:

$$\psi_g^{tl} = -\frac{3}{4\pi G}\frac{p_0 p_0'}{\rho_0^2}r^{l+2t-2} \tag{A18}$$

This is actually a crude solution to the differential equation governing the g-type Lagrangian displacements. The ansatz has the proper behavior, required by the differential equation, at the center and satisfies the proper boundary conditions at the surface of the fluid. Numerical computations of Sobouti (1977b) also confirm that the ansatz of Equation (A18) is capable of locating the g-modes systematically and in a rapidly convergent manner.

On using Equations (7a), (A18), (A3b), and the Lane-Emden equation governing ρ_0 (to eliminate the second derivative of ρ_0, wherever occurring), the following asymptotic behaviors of the diagonal elements of R have been found:

$$R_{0gg}^{tk,sk} \to \int \rho_0\psi_g^{tk}\psi_g^{sk}\frac{dr}{r^2} \tag{A19}$$

as $k \to \infty$, but s and t remain finite;

$$R_{0gg}^{tk,sk} \to \frac{4(2k^2+2k-3)ts}{(2k-1)(2k+3)k(k+1)}\int \rho_0\psi_g^{tk}\psi_g^{sk}\frac{dr}{r^2} \tag{A20}$$

as t and $s \to \infty$, but k remains finite. Reductions leading to Equations (A19) and (A20) are straightforward but elaborate.

Appendix B: Reduction of Equations (8c) and (14)

a) *Closure Relation*

The inner product of two displacements $\boldsymbol{\xi}$ and $\boldsymbol{\xi}'$ of the fluid is suggested by Equation (14c)

$$(\boldsymbol{\xi}, \rho\boldsymbol{\xi}') = \int \rho\boldsymbol{\xi}^*\cdot\boldsymbol{\xi}'dv. \tag{B1}$$

Let $\{\boldsymbol{\zeta}^s\}$ be a complete orthonormal set in some subspace of the Hilbert space to which the Lagrangian displacements of the fluid belong. Thus,

$$(\boldsymbol{\zeta}^r, \rho\boldsymbol{\zeta}^s) = \delta_{rs}. \tag{B2}$$

To obtain the closure relation for $\{\boldsymbol{\zeta}^r\}$, expand an arbitrary $\boldsymbol{\xi}$ in terms of this basis set

$$\boldsymbol{\xi}(\mathbf{r}) = \sum_s \boldsymbol{\zeta}^s(\mathbf{r})(\boldsymbol{\zeta}^s, \rho\boldsymbol{\xi}), \tag{B3}$$

where s could be a discrete and/or continuous label. The inner products, $(\boldsymbol{\zeta}^s, \rho\boldsymbol{\xi})$ are the expansion constants. Expressing these constants in their explicit form of Equation (B1) and requiring that Equation (B3) should be an identity for each three-dimensional space component of $\boldsymbol{\xi}$ and at all points $\mathbf{r}$ leads to the closure relation

$$\sum_s \zeta_i^s(\mathbf{r})\rho(\mathbf{r}')\zeta_j^s(\mathbf{r}') = \delta_{ij}\delta(\mathbf{r}-\mathbf{r}'), \tag{B4}$$

where $i, j = 1, 2, 3$ denote space components of the vectors in some coordinate system.

b) *Reduction of Equation (14)*

Let the basis set $\{\zeta_i^s\}$, for the toroidal subspace of the Lagrangian displacements, be orthonormal. The (pq) element of the Coriolis term in Equation (14) becomes

$$\Omega^2(C_{st}S_{tt}^{-1}C_{ts})^{pq} = \Omega^2\sum_r C_{st}^{pr}C_{ts}^{rq}. \tag{B5}$$

With the assumption of orthonormality, S_{tt} is a unit matrix. Substituting for the C-matrix from Equation (4b), and rearranging the order of integrations over the space coordinates and the summation gives

$$\Omega^2\sum_r C_{st}^{pr}C_{ts}^{rq} = \varepsilon_{ijk}\varepsilon_{lmn}\Omega_j\Omega_m$$

$$\int\int \rho(\mathbf{r})\zeta_{si}^{p*}\left[\sum_r \zeta_{tk}^r(\mathbf{r})\rho(\mathbf{r}')\zeta_{tl}^{r*}(\mathbf{r}')\right]\zeta_{sn}^q(\mathbf{r}')dv\,dv', \tag{B6}$$

where ε_{ijk} is Levi-Civita symbols. Equation (B6) with the closure relation (B4) reduces to

$$\Omega^2 \sum_r C_{\varepsilon t}^{pr} C_{t\varepsilon}^{rq} = -\int \rho(\mathbf{\Omega} \times \boldsymbol{\zeta}_\varepsilon^{p*})\cdot(\mathbf{\Omega} \times \boldsymbol{\zeta}_\varepsilon^{q})dv = -\Omega^2 R_{\varepsilon\varepsilon}^{pq}. \tag{B7}$$

Equation (B7) is employed in the reduction of Equation (14) to (14a).

c) Reduction of Equation (8c)

From Equations (8c) and (13), the ith space-component of the toroidal term of an eigendisplacement $\boldsymbol{\xi}_\varepsilon^s$, is

$$\sum_q \zeta_{ti}^q(\mathbf{r}) Z_{t\varepsilon}^{qs} = \frac{2i\Omega}{\sqrt{\varepsilon^s}} \sum_q \zeta_{ti}^q(\mathbf{r})(S_{tt}^{-1} C_{t\varepsilon} Z_{\varepsilon\varepsilon})^{qs}. \tag{B8}$$

By the orthonormality of $\{\zeta_t\}$, Equation (B2), S_{tt} is a unit matrix. Substituting for $C_{t\varepsilon}$ from Equation (4a) and interchanging the order of integration over the space coordinates and the summation over q, gives

$$\sum_q \zeta_{ti}^q(\mathbf{r}) Z_{t\varepsilon}^{qs} = \frac{2i}{\sqrt{\varepsilon^s}} \varepsilon_{jkl}\Omega_k \times \int \left[\sum_q \zeta_{ti}^q(\mathbf{r})\rho(\mathbf{r}')\zeta_{tj}^q(\mathbf{r}')\right] \zeta_{\varepsilon l}^p(\mathbf{r}') Z_{\varepsilon\varepsilon}^{ps} dv' = \frac{2i}{\sqrt{\varepsilon^s}} \varepsilon_{ikl}\Omega_k \zeta_{\varepsilon l} Z_{\varepsilon\varepsilon}^{ps} \tag{B9}$$

or

$$\sum_q \zeta_t^q Z_{t\varepsilon}^{qs} = \frac{2i}{\sqrt{\varepsilon^s}} \mathbf{\Omega} \times \sum_p \zeta_\varepsilon^p Z_{\varepsilon\varepsilon}^{ps}. \tag{B10}$$

Equation (B10) reduces Equation (8c) to (15).

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