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ASTRONOMY AND ASTROPHYSICS

Linear oscillations of isotropic stellar systems

I. Basic theoretical considerations*

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Summary. Linear perturbations of stellar systems with step-like distributions of the form $F(E)H(-E)$ is studied, where E is the energy integral, and H is Heaviside's step-function. If (a) $dF/dE>0$ and (b) $F|dF/dE|^{-1/2}=0$ at $E=0$, then the operator generating the self gravitation term in Antonov's equation is positive. Then the system is stable against all infinitesimal perturbations of the Boltzmann-Liouville equation. Polytropes with index $0.5<n<1.5$ satisfy conditions a and b and are stable. If a is satisfied and b is not satisfied, instability is possible. Polytropes with $n<0.5$ are examples of such systems. If $dF/dE<0$, the operator generating the self gravitation term is negative and contributes negatively to the stability of the system.

Antonov's equation is cast into a form involving functions of space coordinates only. In the first approximation the perturbation equation is very much similar to the one governing linear motions of a gaseous body. The first order equation is solved for radial oscillations of polytropic stellar systems with $n>0.5$. Volume and surface density perturbations associated with the first few radial modes of some of the models are given.

Key words: stellar systems: stability, normal modes – Antonov's equation – Liouville's equation – density waves

I. Introduction

To understand the stability of star clusters investigators have often resorted to the linearized collisionless Boltzmann-Liouville equation for the answer. Antonov (1962) pioneered on the subject by showing that the linear perturbations of this equation are governed by a self adjoint operator on a six dimensional phase space. Lynden-Bell (1966), Milder (1967), and Lynden-Bell and Sanitt (1969) expanded on the analogy between a star cluster and a corresponding barotropic gas sphere. Lynden-Bell and Milder derived a sufficient condition for stability. The criterion, however, did not prove to be powerful and presumably motivated Lynden-Bell (1969) to develop his Hartree-Fock exchange-operator approach. Ipser and Thorne (1968), primarily interested in the stability of general relativistic star clusters, generalized Antonov's results to the latter cases. All these stability investigations are of a formal nature with few or no applications to specific models.

A French school has arrived at a number of concrete conclusions. Dorémus et al. (1970, 1971) show that the isotropic phase space densities that are constant or decreasing functions of energy are stable against radial perturbations. Dorémus and Feix (1973) and Gillon et al. (1976b) conclude that anisotropic phase densities, F, that are functions of the energy, E, and the magnitude of the angular momentum, J, are stable against radial perturbations, provided $\partial F/\partial E<0$. Gillon et al. (1976a) conclude stability against non-radial perturbations, provided $\partial F/\partial E<0$ and $\partial F/\partial J<0$. In all these papers the authors use the single or multiple or continuously varying water-bag models.

All the works cited above deal with the question of stability. Due to many-dimensionality of the phase space, practically no one has attempted to solve the actual perturbed Liouville equation for a realistic or semi-realistic stellar model. The author is aware of Dorémus and Feix (1972) and of Dorémus and Baumann (1974) who consider eigen-solutions of one dimensional double water-bags.

In Sect. II the background material is summarized. It is shown that the operator giving the perturbation of the self-gravitation is the product of a positive operator and the sign of dF/dE, the derivative of the equilibrium distribution function with respect to the energy. In Sect. III some mathematical preparations are made for the remainder of the paper. In Sect. IV perturbations of the distribution function are expanded in powers of the velocity variable, integrations over the velocity space are carried out, and Antonov's equation is expressed in terms of functions of space coordinates only. In Sect. V the first order eigenvalue equation and the radial eigenvalue equation are abstracted. In Sect. VI radial oscillations of polytropic stellar systems are solved and the corresponding volume and surface density perturbations are calculated. The application to polytropes here is a sample application to illustrate the usage of the formalism developed in this paper. Radial oscillations of other cluster models, non-radial oscillations of stellar systems, and oscillations in the second order of approximation will appear elsewhere.

II. Antonov's equation

Let a stellar system consist of a finite number of stars of unit mass. In a stationary state the system will be assumed to have a distribution function $F(E)H(-E)$, where

$$E=\tfrac{1}{2}v^2+U(\boldsymbol{x}), \qquad U(\boldsymbol{x})=-G\int F(\boldsymbol{x}',\boldsymbol{v}')|\boldsymbol{x}-\boldsymbol{x}'|^{-1}d\boldsymbol{x}'\,d\boldsymbol{v}',$$

and $H(-E)$ is the Heaviside step function, $H(-E)=1$ if $E<0$ and $H(-E)=0$ if $E>0$. The step function ensures vanishing of the distribution function beyond the escape velocity and confines the integrations over the velocity space into a domain $v<v_e$, where $v_e(\boldsymbol{x})=\sqrt{-2U}$ is the escape velocity from the point $\boldsymbol{x}$. Let the

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system undergo an infinitesimal perturbation, $F(E) \to F(E) + \phi(\boldsymbol{x}, \boldsymbol{v}, t)$. It will be assumed that $|\phi(\boldsymbol{x}, \boldsymbol{v}, t)| \ll F(E)$ for all $\boldsymbol{x}$, $\boldsymbol{v}$, and t. This condition, in particular, implies that in the perturbed configuration still $v < v_e$. The unperturbed distribution function satisfies the time-independent Boltzmann-Liouville equation, $DF = 0$. The perturbed distribution satisfies the time-dependent equation, $\partial(F+\phi)/\partial t + D(F+\phi) = 0$. The perturbation ϕ satisfies the linearized Boltzmann-Liouville equation:

$$\frac{\partial \phi}{\partial t} + D\phi - \left[F_E - \frac{1}{2} F\delta(-E)\right] D\delta U = 0, \tag{1}$$

where

$$D = v_i \frac{\partial}{\partial x_i} - \frac{\partial U}{\partial x_i}\frac{\partial}{\partial v_i}, \tag{1a}$$

$$F_E = \frac{dF}{dE}, \qquad -\frac{1}{2}\delta(-E) = \frac{dH(-E)}{dE}, \tag{1b}$$

$$\delta U = -G \int \phi(\boldsymbol{x}', \boldsymbol{v}' t)|\boldsymbol{x} - \boldsymbol{x}'|^{-1} d\tau', \qquad d\tau' = d\boldsymbol{x}' d\boldsymbol{v}'. \tag{1c}$$

Antonov separates ϕ into symmetric and antisymmetric components in $\boldsymbol{v}$:

$$\phi(\boldsymbol{x}, \boldsymbol{v}, t) = \phi_+(\boldsymbol{x}, \boldsymbol{v}, t) + \phi_-(\boldsymbol{x}, \boldsymbol{v}, t). \tag{2}$$

Noting that D is antisymmetric in $\boldsymbol{v}$, Eq. (1) separates correspondingly:

$$\frac{\partial \phi_+}{\partial t} + D\phi_- = 0, \tag{3a}$$

$$\frac{\partial \phi_-}{\partial t} + D\phi_+ + G\left[F_E - \frac{1}{2}F\delta(-E)\right] D \int \phi_+(\boldsymbol{x}', \boldsymbol{v}', t)|\boldsymbol{x} - \boldsymbol{x}'|^{-1} d\tau' = 0. \tag{3b}$$

Eliminating ϕ_+ between the two equations gives:

$$\frac{\partial^2 \phi_-}{\partial t^2} - D^2\phi_- - G\left[F_E - \frac{1}{2}F\delta(-E)\right] D \int D'\phi'_- |\boldsymbol{x} - \boldsymbol{x}'|^{-1} d\tau' = 0. \tag{4}$$

where a prime on an operator or a function means that the operator or the function in question is to be evaluated at $\boldsymbol{x}'$, $\boldsymbol{v}'$. We have also made use of the fact that $Dg(E) = 0$ for any arbitrary g.

Equation (4), when δ-function is discarded and the remainder is divided by F_E, is Antonov's equation; and in that form the operations on ϕ_- are self adjoint. We shall, however, retain the δ-function and employ a slightly different form of the equation. Furthermore, we will consider cases where F_E is either positive, zero, or negative for all E. Let:

$$F_E = \mathrm{sign}(F_E)|F_E|, \qquad \mathrm{sign}(F_E) = \pm 1, \tag{5a}$$

$$\phi_-(\boldsymbol{x}, \boldsymbol{v}, t) = |F_E|^{1/2} f(\boldsymbol{x}, \boldsymbol{v}, t). \tag{5b}$$

We note that f is antisymmetric in $\boldsymbol{v}$, $f(\boldsymbol{x}, \boldsymbol{v}, t) = -f(\boldsymbol{x}, -\boldsymbol{v}, t)$. Substituting Eqs. (5) in Eq. (4) and dividing by $|F_E|^{1/2}$ gives:

$$\frac{\partial^2 f}{\partial t^2} + \mathscr{W} f + \frac{1}{2} G\,\mathrm{sign}(F_E) F |F_E|^{-1/2} \cdot \delta(-E) D \int |F'_E|^{1/2} D' f' |\boldsymbol{x} - \boldsymbol{x}'|^{-1} d\tau' = 0, \tag{6}$$

where

$$\mathscr{W} = \mathscr{W}_1 + \mathrm{sign}(F_E)\mathscr{W}_2, \tag{6a}$$

$$\mathscr{W}_1 f = -D^2 f, \tag{6b}$$

$$\mathscr{W}_2 f = -G|F_E|^{1/2} D \int |F'_E|^{1/2} D' f' |\boldsymbol{x} - \boldsymbol{x}'|^{-1} d\tau'. \tag{6c}$$

The operators $\mathscr{W}$, $\mathscr{W}_1$, and $\mathscr{W}_2$ are all symmetric, in the sense that $\int g^* \mathscr{W} h d\tau = \int (\mathscr{W} g^*) h d\tau$ for any bounded functions g and h on the phase space. This in turn follows from the fact that D is real antisymmetric, in the sense that $\int g^* D h d\tau = -\int (Dg^*) h d\tau$.

The operation containing the δ-function in Eq. (6) is not symmetric. If, however, $F|F_E|^{-1/2} = 0$ at $E = 0$ (*mild* discontinuity), this term vanishes and Eq. (6) becomes an eigenvalue problem for the symmetric operator $\mathscr{W}$. In the remainder of this paper we shall mainly deal with mildly discontinuous distributions. For the standing wave solutions of such systems, $f = f(\boldsymbol{x}, \boldsymbol{v})\exp(i\omega t)$, Eq. (6) becomes

$$\omega^2 f = \mathscr{W} f. \tag{7}$$

The symmetry of $\mathscr{W}$-operator ensures that the eigenvalues, ω^2, are real and any two eigenfunctions f_i and f_j belonging to two distinct eigenvalues are orthogonal, in the sense that $\int f_i^* f_j d\tau = 0$.

In the case of severe discontinuities, $F|F_E|^{-1/2} \neq 0$ at $E = 0$, the operation on f in Eq. (6) contains the non-symmetric δ-function component. The possibility of complex eigenvalues and, therefore, the possibility of (perhaps vibrational) instability is not ruled out. Polytropes with index $n < 0.5$ are examples of severe discontinuities. Polytropes with $0.5 < n < 1.5$ and $1.5 < n$ are examples of mild discontinuities. The intermediate polytrope $n = 1.5$ has $F = $ constant and $F_E = 0$ (see Sect. VI). The transformation of Eq. (5b) and, therefore, Eq. (6) are not valid for polytrope 1.5. We shall come back to this shortly below.

One may obtain a variational expression for ω^2 by left-multiplying Eq. (7) by f^* and integrating over the phase space. Thus,

$$\omega^2 = [W_1 + \mathrm{sign}(F_E) W_2]/S, \tag{8}$$

where

$$S = \int f^* f d\tau > 0, \tag{8a}$$

$$W_1 = \int f^* \mathscr{W}_1 f d\tau = \int Df^* Df d\tau \geqq 0, \tag{8b}$$

$$W_2 = \int f^* \mathscr{W}_2 f d\tau = G \iint |F_E|^{1/2} |F'_E|^{1/2} \cdot Df^* D'f' |\boldsymbol{x} - \boldsymbol{x}'|^{-1} d\tau\, d\tau' \geqq 0. \tag{8c}$$

The S-integral is positive definite for $f \neq 0$. The W_1-integral and therefore the $\mathscr{W}_1$-operator is positive. They can be zero if and only if $Df = 0$ throughout the phase space. These properties are evident from Eqs. (8a) and (8b). That the W_2-integral and, therefore, the $\mathscr{W}_2$-operator are also positive is shown below.

Theorem. The W_2-integral and consequently the $\mathscr{W}_2$-operator are positive.

Proof. From Eqs. (5b), (3a), and the exponential time dependence of f one has $\int |F_E|^{1/2} Df d\boldsymbol{v} = i\omega\delta\varrho$, where $\delta\varrho = \int \phi_+ d\boldsymbol{v}$ is the perturbation in the mass density. Similarly,

$$-G \int |F'_E|^{1/2} D' f' |\boldsymbol{x} - \boldsymbol{x}'|^{-1} d\tau' = -i\omega G \int \delta\varrho(\boldsymbol{x}')|\boldsymbol{x} - \boldsymbol{x}'|^{-1} d\boldsymbol{x}' = i\omega\delta U,$$

where δU is the perturbation in the gravitational potential. It satisfies the Poisson equation $\nabla^2\delta U = 4\pi G\delta\varrho$. In terms of $\delta\varrho$ and δU, one now has

$$W_2 = -|\omega|^2 \int_{\text{all space}} \delta\varrho^* \delta U d\boldsymbol{x} = -\frac{|\omega|^2}{4\pi G} \int_{\text{all space}} \nabla^2 \delta U^* \delta U d\boldsymbol{x}. \tag{9a}$$

Extension of the domain of integration over all space is permissible, since $\delta\varrho = 0$ outside the volume of the stellar system. On integrating Eq. (9a) by parts and letting the integrated part vanish at infinity, one obtains:

$$W_2 = \frac{|\omega|^2}{4\pi G} \int_{\text{all space}} \nabla(\delta U^*)\nabla \cdot (\delta U) d\boldsymbol{x} > 0\,, \tag{9b}$$

where again we observe that $W_2 = 0$ if and only if $Df = 0$. Q.E.D.

An alternative and entirely different proof of the theorem is given by Sobouti [1980, Eqs. (9.10)]. There, the author gives a spherical harmonic expansion of W_2 and shows that every term in the expansion is positive.

We are now ready to announce the following:

Theorem. A system with the equilibrium distribution of the form $F(E)H(-E)$, in which F is an increasing function of $E[\text{sign}(F_E) = 1]$ and $F|F_E|^{-1/2} = 0$ at $E = 0$, is stable against infinitesimal perturbations of the collisionless Boltzmann-Liouville equation.

The proof is evident from Eq. (8) and the positive character of the W_1-, W_2-, and S-integrals. Restriction to $F|F_E|^{-1/2} = 0$ at $E = 0$ is to ensure that the negative δ-function singularities at $E = 0$ can be discarded.

It is remarkable that all the works cited in Sect. I consider $dF/dE < 0$ as a condition for stability. What the present analysis shows is to the contrary. A positive energy gradient of F has a stabilizing effect; and a negative gradient lowers the eigenfrequencies and makes the system less stable. Thus, the polytropes with index $0.5 < n < 1.5$ satisfy the conditions of the theorem above and are stable to all infinitesimal perturbations (see Sect. VI). Henon's (1972) numerical experiments on polytropes $0.5 < n < 1.5$ are in complete agreement with the present conclusions.

Lynden-Bell (1967), discussing homogeneous media, uses distribution functions of the type $F + M$, where M is a constant step function. He finds instability for certain range of the data. Although there is no strict parallelism between Lynden-Bell's problem and the possibility of instability in our severely discontinuous step-like distributions, there may be a kinship between the two problems.

Constant step function distribution

For the polytrope 1.5, the distribution is $H(-E)$ with the energy derivative $\frac{1}{2}\delta(-E)$. The transformation of Eq. (5b) and consequently Eqs. (6)–(8) are not valid (for $F_E = 0$). However, multiplying Eq. (4) by ϕ_-^* and integrating over the phase space yields:

$$\omega^2 \int \phi_-^* \phi_- d\tau = \int D\phi_-^* D\phi_- d\tau$$
$$-\tfrac{1}{2} G \int (D\phi_-)_{v=v_e} d\boldsymbol{x} \int D'\phi'_- |\boldsymbol{x} - \boldsymbol{x}|^{-1} d\tau'\,.$$

The term $D\phi_- = \partial\phi_+/\partial t$ vanishes at the escape velocity. What remains is an eigenvalue equation with positive real ω^2. Thus, constant step distributions are stable. This derivation is an alternative to the derivation of Dorémus et al. (1971) who employ their water bag technique. The numerical computations of Sect. VI for $n = 1.5$ are solutions of ϕ_- of the above equation (with the third term omitted) rather than the solution f of Eqs. (8).

III. Moments of functions of energy

This section is devoted to certain preliminaries for manipulating the eigenvalue Eq. (7). First we introduce a set of constant tensors which have been found useful in the study of the moments of functions which depend on the magnitude of the velocity but not on its direction.

The α-symbols

A set of completely symmetric constant tensors of rank n, $\alpha^{(n)}_{i_1 \ldots i_n}$, $n = 0, 2, 4, \ldots$, are defined by the following recursion relation

$$\alpha^{(n)}_{i_1 i_2 \ldots i_n} = \delta_{i_1 i_2} \alpha^{(n-2)}_{i_3 \ldots i_n} + \ldots + \delta_{i_1 i_n} \alpha^{(n-2)}_{i_2 \ldots i_{n-1}}\,, \tag{10a}$$

$$\alpha^{(0)} = 1\,, \tag{10b}$$

where δ_{ij} is the Kronecker delta. Contraction of $\alpha^{(n)}$ over a pair of indices gives

$$\alpha^{(n)}_{i_1 i_1 i_3 \ldots i_n} = (n+1)\alpha^{(n-2)}_{i_3 \ldots i_n}\,. \tag{11}$$

Examples of the α-symbols are:

$$\alpha^{(2)}_{ij} = \delta_{ij}\,, \tag{12a}$$

$$\alpha^{(4)}_{ijkl} = \delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\,. \tag{12b}$$

The α-symbols of odd rank are zero.

Integration over space directions

Let $\boldsymbol{u}$ be a unit vector in a (θ, ϕ)-direction. Its direction-cosines are $u_1 = \sin\theta \cos\phi$, $u_2 = \sin\theta \sin\phi$, and $u_3 = \cos\theta$. The integral of the product of n such direction-cosines over space directions is given in terms of the α-symbols. Thus,

$$\iint u_{i_1} \ldots u_{i_n} \sin\theta\, d\theta\, d\varphi = \frac{4\pi}{1.3 \ldots (n+1)} \alpha^{(n)}_{i_1 \ldots i_n}\,, \qquad n = 0, 2, \ldots\,, \tag{13}$$

Examples of Eq. (13) are

$$\int \sin\theta\, d\theta\, d\varphi = 4\pi\,, \qquad \int u_i u_j \sin\theta\, d\theta\, d\varphi = \frac{4\pi}{3}\delta_{ij}\,,$$

etc. Note that the integral is non-zero if the indices $i_1, \ldots, i_n$ are pairwise the same. Otherwise it vanishes.

Moments of functions of energy

Let $H(E)$ be any function of $E = \frac{1}{2}v^2 + U(\boldsymbol{x})$. Define the n^{th} moment of H as follows

$$H^{(n)}_{i_1 \ldots i_n}(U) = \int H(E) v_{i_1} \ldots v_{i_n} d\boldsymbol{v}\,. \tag{14}$$

Let $v_i = v u_i$, where v is the magnitude of the velocity vector and u_i is its i^{th} direction-cosine. Substituting this in Eq. (14) and using Eq. (13) gives:

$$H^{(n)}_{i_1 \ldots i_n}(U) = H^{(n)}(U)\alpha^{(n)}_{i_1 \ldots i_n}\,, \tag{15a}$$

where

$$H^{(n)}(U) = \frac{4\pi}{1.3 \ldots (n+1)} \int_0^{\sqrt{-2U}} H(E) v^{n+2} dv\,. \tag{15b}$$

The following recursion relation exists for $H^{(n)}(U)$:

$$\frac{dH^{(n)}}{dU} = -H^{(n-2)}\,, \tag{15c}$$

or

$$\nabla H^n = -\nabla U H^{(n-2)}\,. \tag{15d}$$

Equation (15c) can be verified by direct differentiation of the defining Eq. (15b).

The moments of $|F_E|^{1/2}$ and 1 will be encountered frequently in the remainder of this paper. The following notation is reserved for these moments:

$$\Psi^{(n)}=\frac{4\pi}{1.3\ldots(n+1)}\int_0^{\sqrt{-2U}}|F_E|^{1/2}v^{n+2}dv\,, \tag{16a}$$

$$\Phi^{(n)}=\frac{4\pi}{1.3\ldots(n+1)}\int_0^{\sqrt{-2U}}1\cdot v^{n+2}dv =\frac{4\pi}{1.3\ldots(n+3)}(-2U)^{(n+3)/2}\,. \tag{16b}$$

Of particular interest are the moments of the equilibrium distribution function, $F(E)$. Let

$$\Gamma^{(n)}(U)=\frac{4\pi}{1.3\ldots(n+1)}\int_0^{\sqrt{-2U}}F(E)v^{n+2}dv\,. \tag{17}$$

Examples of $\Gamma^{(n)}$ are

$$\Gamma^{(0)}=\varrho=4\pi\int F(E)v^2dv$$

and

$$\Gamma^{(2)}=p=(4\pi/3)\int F(E)v^4dv\,,$$

where ϱ and p are the conventional density and pressure of the stellar configuration, respectively. According to Eq. (15d) they satisfy the following:

$$\nabla p=-\varrho\nabla U\,. \tag{18}$$

Thus, the celebrated equation of hydrostatic equilibrium emerges as a special case of a far more general relation, Eq. (15d).

IV. Expansion of functions in terms of the moments of their Fourier transforms

The perturbation distribution function $f(\boldsymbol{x},\boldsymbol{v})$ is antisymmetric in $\boldsymbol{v}$. Let $\xi(\boldsymbol{x},\boldsymbol{k})$ be its Fourier transform with respect to $\boldsymbol{v}$,

$$f(\boldsymbol{x},\boldsymbol{v})=\int\xi(\boldsymbol{x},\boldsymbol{k})\sin(k_iv_i)d\boldsymbol{k}\,. \tag{19a}$$

One has the expansion

$$\sin(k_iv_i)\cdot\sum_n(-1)^{(n-1)/2}(k_iv_i)^n/n! =\sum(-1)^{(n-1)/2}k_{i_1}\ldots k_{i_n}v_{i_1}\ldots v_{i_n}/n!\,,$$

where the summation is over odd values of n. Substituting this expansion in Eq. (19a) and carrying out the integration over the k-space gives:

$$f(\boldsymbol{x},\boldsymbol{v})=\sum_n\frac{(-1)^{(n-1)/2}}{n!}\xi^{(n)}_{i_1\ldots i_n}v_{i_1}\ldots v_{i_n}\,,\qquad n=\text{odd}\,, \tag{19b}$$

where $\xi^{(n)}_{i_1\ldots i_n}(\boldsymbol{x})$ is the n^{th} moment of $\xi(\boldsymbol{x},\boldsymbol{k})$ in the k-space:

$$\xi^{(n)}_{i_1\ldots i_n}(\boldsymbol{x})=\int\xi(\boldsymbol{x},\boldsymbol{k})k_{i_1}\ldots k_{i_n}d\boldsymbol{k}\,,\qquad n=\text{odd}\,. \tag{19c}$$

Equation (19b) is an expansion of $f(\boldsymbol{x},\boldsymbol{v})$ as a power series in v_i. Dependence on $\boldsymbol{x}$ now appears in $\xi^{(n)}_{i_1\ldots i_1}$, the moments of the Fourier transform of $f(\boldsymbol{x},\boldsymbol{v})$. From the inverse Fourier transformation one has:

$$\xi(\boldsymbol{x},\boldsymbol{k})=\frac{8}{\pi^3}\int f(\boldsymbol{x},\boldsymbol{v})\sin(k_jv_j)d\boldsymbol{v}\,, \tag{20a}$$

$$\xi(\boldsymbol{x},\boldsymbol{k})=\frac{8}{\pi^3}\sum_m\frac{(-1)^{(m-1)/2}}{m!}f_{j_1\ldots j_m}(\boldsymbol{x})k_{j_1}\ldots k_{j_m}\,, \tag{20b}$$

$$f^{(m)}_{j_1\ldots j_m}(\boldsymbol{x})=\int f(\boldsymbol{x},\boldsymbol{v})v_{j_1}\ldots v_{j_m}d\boldsymbol{v}\,,\qquad m=\text{odd}\,. \tag{20c}$$

We observe that $\xi(\boldsymbol{x},\boldsymbol{k})$ is antisymmetric in $\boldsymbol{k}$. Corresponding to the limiting velocity, $v_e=\sqrt{(-2U)}$, there is a limiting value for k, $k_e=\pi/v_e$, which is the first zero of $\sin(k_ev_e)=0$.

One may go one step further and express $f^{(m)}$ in terms $\xi^{(n)}$ and vice versa. Substituting Eq. (19a) in Eq. (20c) and carrying out integration over $\boldsymbol{v}$ by means of Eqs. (14), (15), and (16b) gives

$$f^{(m)}_{j_1\ldots j_m}(\boldsymbol{x})=\sum_n\frac{(-1)^{(n-1)/2}}{n!}\alpha^{(m+n)}_{j_1\ldots j_m i_1\ldots i_n}\xi^{(n)}_{i_1\ldots i_n}\Phi^{(m+n)}(U)\,. \tag{21a}$$

Inversely:

$$\xi^{(n)}_{i_1\ldots i_n}(\boldsymbol{x})=\frac{8}{\pi^3}\sum_m\frac{(-1)^{(m-1)/2}}{m!}\alpha^{(n+m)}_{i_1\ldots i_n j_1\ldots j_m}f^{(m)}_{j_1\ldots j_m}\Phi^{(n+m)}(-\pi^2/2U)\,, \tag{21b}$$

where $\Phi^{(j)}(U)$ is given by Eq. (16b). With the aid of Eqs. (19)–(21) the other quantities can now be expanded.

Expansion of Df

Letting $D=v_i\frac{\partial}{\partial x_i}-\frac{\partial U}{\partial x_i}\frac{\partial}{\partial v_i}$ operate on Eq. (19b) gives:

$$Df=\sum_n\frac{(-1)^{(n-1)/2}}{n!}\left\{\frac{\partial\xi^{(n)}_{i_1\ldots i_n}}{\partial x_{i_{n+1}}}v_{i_1}\ldots v_{i_n}-\frac{\partial U}{\partial x_{i_{n+1}}}\xi^{(n)}_{i_1\ldots i_n}\cdot[\delta_{i_1i_{n+1}}v_{i_2}\ldots v_{i_n}+\ldots+\delta_{i_ni_{n+1}}v_{i_1}\ldots v_{i_{n-1}}]\right\}. \tag{22}$$

The S-integral

Substituting Eq. (19a) in Eq. (8a), integrating over the directions of the velocity vector by Eq. (13), and using Eq. (16b) gives:

$$S=-\sum_{m,n}\frac{(-1)^{(m+n)/2}}{m!n!}\alpha^{(m+n)}_{i_1\ldots i_m j_1\ldots j_n}\int\Phi^{(m+n)}\xi^{(m)}_{i_1\ldots i_m}\xi^{(n)}_{j_1\ldots j_n}d\boldsymbol{x}\,,$$

$$m,n=\text{odd integers}. \tag{23}$$

The W_1-integral

The expression for this integral is lengthy, but straightforward to obtain. Substituting Eq. (22) in Eq. (8b) and integrating over the velocity space, as in S, gives:

$$\begin{aligned}W_1=\sum_{n,m}\Bigg\{&A_{i_1\ldots i_{m+1},j_1\ldots j_{n+1}}\int\frac{\partial\xi^{(m)}_{i_1\ldots i_m}}{\partial x_{i_{m+1}}}\frac{\partial\xi^{(n)}_{j_1\ldots j_n}}{\partial x_{j_{n+1}}}\Phi^{(m+n+2)}d\boldsymbol{x}\\&+B_{i_1\ldots i_{m+1},j_1\ldots j_{n+1}}\int\xi^{(m)}_{i_1\ldots i_m}\frac{\partial\xi^{(n)}_{j_1\ldots j_n}}{\partial x_{j_{n+1}}}\frac{\partial U}{\partial x_{im+1}}\Phi^{(m+n)}d\boldsymbol{x}\\&+C_{i_1\ldots i_{m+1},j_1\ldots j_{n+1}}\int\xi^{(m)}_{i_1\ldots i_m}\xi^{(n)}_{j_1\ldots j_n}\cdot\frac{\partial U}{\partial x_{i_{m+1}}}\frac{\partial U}{\partial x_{j_{n+1}}}\Phi^{(m+n-2)}d\boldsymbol{x}\Bigg\},\end{aligned} \tag{24}$$

$$A_{i_1\ldots i_{m+1},j_1\ldots j_{n+1}}=-\frac{(-1)^{(m+n)/2}}{m!n!}\alpha^{(m+n+2)}_{i_1\ldots i_{m+1}j_1\ldots j_{n+1}}\,, \tag{24a}$$

$$B_{i_1\ldots i_{m+1},j_1\ldots j_{n+1}}=2\frac{(-1)^{(m+n)/2}}{m!n!}\cdot[\delta_{i_{m+1}i_1}\alpha^{(m+n)}_{i_2\ldots i_m j_1\ldots j_{n+1}}+\text{permutations of } i_1 \text{ to } i_m]\,, \tag{24b}$$

$$C_{i_1\ldots i_{m+1},j_1\ldots j_{n+1}}=-\frac{(-1)^{(m+n)/2}}{m!n!}\cdot\left[\delta_{i_{m+1}i_1}\delta_{j_{n+1}j_i}\alpha^{(m+n-2)}_{i_2\ldots i_m j_2\ldots j_n}+\text{permutations of } i_1 \text{ to } i_m \text{ and of } j_1 \text{ to } j_m\right]. \tag{24c}$$

The W_2-integral

The integral $\int |F_E|^{1/2} Df dv = i\omega\delta\varrho$ appears in Eq. (8c). Substituting Eq. (22) in this integral, and using Eqs. (13), (10a), and (16b) gives:

$$i\omega\delta\varrho=\int |F_E|^{1/2}Df\,dv$$
$$=\sum_m \frac{(-1)^{(m-1)/2}}{m!}\alpha^{(m+1)}_{i_1\dots i_{m+1}}\frac{\partial}{\partial x_{i_{m+1}}}[\Psi^{(m+1)}\xi^{(m)}_{i_1\dots i_m}]. \tag{25}$$

Substituting Eq. (5) in Eq. (8c) gives:

$$W_2=-G\sum_{m,n}\frac{(-1)^{(m+n)/2}}{m!n!}\alpha^{(m+1)}_{i_1\dots i_{m+1}}\alpha^{(n+1)}_{j_1\dots j_{n+1}}$$
$$\cdot\int\frac{\partial}{\partial x_{i_{m+1}}}[\Psi^{(m+1)}\xi^{(m)}_{i_1\dots i_m}]\frac{\partial}{\partial x'_{j_{n+1}}}[\Psi^{(n+1)}\xi_{j_1\dots j_n}]'\frac{dx\,dx'}{|x-x'|}. \tag{26}$$

where the "prime" on the bracket indicates that it is to be evaluated at x'.

The main features of the formalism are outlined. The next step is to attempt solution of Eqs. (8) and (23)–(24) in successive approximations. Equations (23)–(26) involve integrations over the x-space, only. Difficulties of dealing with the full six dimensional phase-space, and along with it some of the conceptual obscurities are removed. In the first approximation close similarities with the linear oscillations of gas spheres have been detected. Some of these similarities will be pointed out and utilized below when we discuss radial oscillations.

Remarks on truncated series and operators

The perturbations $f(x, v)$ of Eqs. (6)–(8) belong to a Hilbert space $\mathscr{H}=\mathscr{H}_x\otimes\mathscr{H}_v$ where $\mathscr{H}_x$ and $\mathscr{H}_v$ denote the two components of $\mathscr{H}$ in the configuration variable x and the velocity variable v, respectively. The operator $\mathscr{W}$ of Eqs. (6) defined on $\mathscr{H}$ is Hermitian. Therefore, its eigenfunctions form a complete set and can serve as a basis for $\mathscr{H}$. Truncation of the series of Eq. (19b) and the subsequent equations at the n^{th} term is equivalent to confining oneself to a subspace $\mathscr{H}(n)=\mathscr{H}_x\otimes\mathscr{H}_v(n)$ of the fuller space $\mathscr{H}$, where $\mathscr{H}_v(n)$ is the vector space of the n^{th} order polynomials in v, and is a n-dimensional subspace of $\mathscr{H}_v$. Let the $\mathscr{W}$ operator defined on $\mathscr{H}(n)$ be denoted by $\mathscr{W}(n)$. This last operator is Hermitian on $\mathscr{H}(n)$ and therefore its eigenfunctions form a complete set in $\mathscr{H}(n)$. One further observation: It was found in Sect. II that the S-integral was positive definite and W_1 and W_2-integrals, or their corresponding operators $\mathscr{W}_1$ and $\mathscr{W}_2$, were positive. Any n^{th} order truncation of these integrals and operators will have the same positive character as the original integrals and operators.

V. The first order eigenvalue problem

Let us repeat Eq. (8) here:

$$\omega^2=\frac{W_1+\text{sign}(F_E)W_2}{S}. \tag{8}$$

Keeping only the first $m=n=1$ terms in Eqs. (24)–(26) gives

$$S=\int\Phi^{(2)}\xi_i\xi_i dx, \tag{27}$$

$$W_1=\int\Phi^{(4)}\left[\frac{\partial\xi_i}{\partial x_i}\frac{\partial\xi_j}{\partial x_j}+\frac{\partial\xi_i}{\partial x_j}\frac{\partial\xi_i}{\partial x_j}+\frac{\partial\xi_i}{\partial x_j}\frac{\partial\xi_j}{\partial x_i}\right]dx$$
$$+2\int\frac{\partial\Phi^{(4)}}{\partial x_i}\xi_i\frac{\partial\xi_j}{\partial x_j}dx+\int\Phi^{(0)}\frac{\partial U}{\partial x_i}\frac{\partial U}{\partial x_j}\xi_i\xi_j dx$$
$$=\int\Phi^{(4)}\left[\frac{\partial\xi_i}{\partial x_j}\frac{\partial\xi_i}{\partial x_j}+2\frac{\partial\xi_i}{\partial x_j}\frac{\partial\xi_j}{\partial x_i}\right]dx+\int\Phi^{(2)}\frac{\partial^2 U}{\partial x_i\partial x_j}\xi_i\xi_j dx, \tag{28}$$

$$W_2=G\int\frac{\partial}{\partial x_i}[\Psi^{(2)}\xi_i]\frac{\partial}{\partial x'_j}[\Psi^{(2)}\xi_j]'|x-x'|^{-1}dx\,dx', \tag{29}$$

where the superscript in $\xi^{(1)}_i$ is suppressed. The second form of W_1 is obtained from the first form by integrating the first and the last terms by parts. The $\xi_i(x)$ is a vector field defined over the volume of the stellar system. In fact, it is an indicator of the macroscopic velocity of the perturbed configuration. From Eq. (21b), keeping only the first term, one has

$$\xi_i(x)=\frac{32\pi^3}{15}(-2U)^{-5/2}f_i(x),$$

where $f_i(x)=\int f(x,v)v_i dv$ is the macroscopic velocity. As pointed out at the end of the last section, the first order S is positive definite, and the first order W_1 and W_2 are positive. These conclusions follow from the corresponding characters of Eqs. (8), which hold for any arbitrary $f(x, v)$. In Eqs. (27)–(29) it is assumed that $f(x, v) =\xi_i(x)v_i$, which is the first term of Eq. (19b).

A differential form of Eqs. (8) and (27)–(29) can also be obtained by requiring ω^2 to be stationary with respect to arbitrary variations $\delta\xi_i(x)$ of $\xi_i(x)$ which vanish at the boundary of the configuration. We skip the derivation and give the final result. Thus,

$$\omega^2\Phi^{(2)}\xi_i=[\mathscr{W}_1+\text{sign}(F_E)\mathscr{W}_2]\xi_i, \tag{30}$$

where

$$\mathscr{W}_1\xi_i=-\alpha^{(4)}_{ijkl}\frac{\partial}{\partial x_j}\left[\Phi^{(4)}\frac{\partial\xi_k}{\partial x_l}\right]$$
$$+\frac{\partial\Phi^{(4)}}{\partial x_i}\frac{\partial\xi_j}{\partial x_j}-\frac{\partial}{\partial x_i}\left[\frac{\partial\Phi^{(4)}}{\partial x_j}\xi_j\right]$$
$$-\frac{\partial\Phi^{(2)}}{\partial x_i}\frac{\partial U}{\partial x_j}\xi_j, \tag{30a}$$

$$\mathscr{W}_2\xi_1=-G\Psi^{(2)}\frac{\partial}{\partial x_i}\int\frac{\partial}{\partial x'_j}[\Psi^{(2)}\xi_j]'\frac{dx'}{|x-x'|}. \tag{30b}$$

The operators $\mathscr{W}_1$ and $\mathscr{W}_2$ are self adjoint. This, which can be verified easily, is a consequence of the self adjoint character of the main operators of Eqs. (6c) and (6d).

The first order density waves

Should perturbations of the type considered here have any relevance to actual stellar systems, then the best candidates for observation is the density perturbation, $\delta\varrho=\int\varphi_+ dv$. Integrating Eq. (3a) over the velocity space, substituting Eq. (25) in the resulting expression, and retaining only the first term of the expansion gives

$$\delta\varrho=-\frac{i}{\omega}\frac{\partial}{\partial x_i}[\Psi^{(2)}\xi_i]. \tag{31}$$

A wave-like behavior of this quantity is termed a *density wave* here. Associated with $\delta\varrho$ is the surface density perturbation. Both volume and surface density perturbations will be calculated for the models considered here.

The first order radial oscillation

In (r, θ, ϕ) coordinates let $\xi_i=[\xi(r), 0, 0]$. To obtain the appropriate S-, W_1-, and W_2-integrals, we found it convenient to write

Eqs. (27)–(29) in a covariant form, and then to work out various covariant derivatives in the spherical coordinates. In this way, for three of the terms in the W_1 integral of Eq. (28) we found

$$\partial\xi_i/\partial x_i = d\xi/dr + 2\xi/r\,,$$
$$(\partial\xi_i/\partial x_j)(\partial\xi_i/\partial x_j) = (\partial\xi_i/\partial x_j)(\partial\xi_j/\partial x_i) = (d\xi/dr)^2 + 2\xi^2/r^2\,.$$

Equation (27) reduces to:

$$S = 4\pi\int_0^R \Phi^{(2)}\xi^2 r^2 dr\,, \qquad (32)$$

where R is the boundary radius of the system. Equation (28) gives:

$$W_1 = 4\pi\left\{\int_0^R \Phi^{(4)}\left[\left(\frac{d\xi}{dr}+\frac{2}{r}\xi\right)^2 + 2\left(\frac{d\xi}{dr}\right)^2 + \frac{4}{r^2}\xi^2\right]r^2 dr + 2\int_0^R \frac{d\Phi^{(4)}}{dr}\xi\left[\frac{d\xi}{dr}+\frac{2}{r}\xi\right]r^2 dr - \int_0^R \frac{d\Phi^{(2)}}{dr}\frac{dU}{dr}\xi^2 r^2 dr\right\}. \qquad (33a)$$

Integrating by parts to eliminate $\xi d\xi/dr$ terms and $d\Phi^{(2)}/dr$, and using the Poisson equation $r^{-2}d(r^2 dU/dr)/dr = 4\pi G\varrho$, and Eq. (15d) reduces Eq. (33a) to the following:

$$W_1 = 4\pi\left\{3\int_0^R \Phi^{(4)}\left(\frac{d\xi}{dr}\right)^2 r^2 dr + 2\int_0^R\left[3\Phi^{(4)} + r\frac{d\Phi^{(4)}}{dr} + 2\pi G\varrho\Phi^{(2)}r^2\right]\xi^2 dr\right\}. \qquad (33b)$$

Equation (29) reduces to:

$$W_2 = G\int \frac{1}{r^2}\frac{d}{dr}[r^2\Psi^{(2)}\xi]\frac{1}{r'^2}\frac{d}{dr'}[r'^2\Psi^{(2)}\xi]'\frac{dx\,dx'}{|x-x'|}. \qquad (34a)$$

Expanding $|x-x'|^{-1}$ in Legendre polynomials and carrying out the angular integration gives:

$$W_2 = 16\pi^2 G\int_0^R [\Psi^{(2)}]^2\xi^2 r^2 dr\,. \qquad (34b)$$

Equation (34b) can also be obtained directly from Eq. (9b). From Eq. (25) and the Poisson equation one has

$$\nabla^2\delta U = -4\pi G(i/\omega)\nabla\cdot[\Psi^{(2)}\xi]\,.$$

For radial perturbations both $\nabla\delta U$ and $\Psi^{(2)}\xi$ vanish at the surface. Therefore, $\nabla\delta U = -4\pi G(i/\omega)\Psi^{(2)}\xi$. Substituting this in Eq. (9b) gives Eq. (34b).

Again a differential equation for radial eigenfunctions can be obtained from Eqs. (32), (34b), and (34a) by a variational technique. Thus,

$$\omega^2\Phi^{(2)}\xi = \mathscr{W}_1\xi + \mathrm{sign}(F_E)\mathscr{W}_2\xi\,, \qquad (35)$$

where

$$\mathscr{W}_1\xi = -3\frac{1}{r^2}\frac{d}{dr}\left[r^2\Phi^{(4)}\frac{d\xi}{dr}\right] + 2\left[\frac{3}{r^2}\Phi^{(4)} + \frac{1}{r}\frac{d\Phi^{(4)}}{dr} + 2\pi G\varrho\Phi^{(2)}\right]\xi\,, \qquad (35a)$$

$$\mathscr{W}_2\xi = -G\Psi^{(2)}\frac{d}{dr}\int\frac{1}{r'^2}\frac{d}{dr'}[r'^2\Psi^{(2)}\xi]'\frac{dx'}{|x-x'|} = 16\pi^2 G[\Psi^{(2)}]^2 r^2\xi\,. \qquad (35b)$$

Again the operators $\mathscr{W}_1$ and $\mathscr{W}_2$ for these radial oscillations are self adjoint.

The corresponding density perturbation is

$$\delta\varrho(r) = -\frac{i}{\omega}\frac{1}{r^2}\frac{d}{dr}[r^2\Psi^{(2)}\xi]\,. \qquad (36a)$$

In a cylindrical coordinate (ϖ,ϕ,z), let $r^2=\varpi^2+z^2$. Integrating Eq. (36a) over the z values gives the surface density perturbations. Thus,

$$\delta\sigma(\varpi) = 2\int_0^{(R^2-\varpi^2)^{1/2}}\delta\varrho\,dz\,. \qquad (36b)$$

The surface density waves will be directly calculated from Eq. (36b).

VI. Application to polytropes

The distribution function for a polytrope of index n can be written as follows (Eddington, 1916):

$$F_n(E) = \frac{\alpha_n}{8\pi\sqrt{2}}(-E)^{n-3/2} \quad \text{for} \quad E=\frac{1}{2}v^2+U<0$$
$$=0 \quad \text{for} \quad E>0 \qquad (37)$$

where α_n is a constant. The corresponding density, $\varrho_n=\int F_n dv$, is:

$$\varrho_n = \alpha_n\beta_n(-U)^n\,, \qquad (38)$$

where β_n is a definite integral which emerges in the course of integration over the velocity space. Thus,

$$\beta_n = \int_0^1 (1-\eta^2)^{n-3/2}\eta^{3/2}d\eta = \frac{\sqrt{\pi}}{2}\frac{\Gamma(n-\frac{1}{2})}{\Gamma(n+1)}, \qquad (39)$$

where $\Gamma(m)$ is a Γ-function. We shall restrict our analysis to $n>\frac{1}{2}$, since $\beta_{\frac{1}{2}}$ as given by Eq. (39) is not defined. In Eqs. (37) and (38) the potential U is chosen zero at the surface of the stellar configuration. Therefore, the velocity of escape defined as $v_e=\sqrt{(-2U)}$ means escape to the boundary of the system rather than to infinity.

To obtain the Lane-Emden equation, let

$$\varrho=\varrho_c\theta^n\,, \quad U=-(\varrho_c/\alpha_n\beta_n)^{1/n}\theta\,, \quad r=\alpha\zeta\,,$$

where ϱ_c is the cnetral density and $\alpha^{-2}=4\pi G\varrho_c(\alpha_n\beta_n/\varrho_c)^{1/n}$. In terms of the polytropic variables, θ and ζ, the Poisson equation becomes:

$$\frac{1}{\zeta^2}\frac{d}{d\zeta}\left(\zeta^2\frac{d\theta}{d\zeta}\right) = -\theta^n\,. \qquad (40)$$

For details of polytropes see Chandrasekhar (1939).

The functions Φ and Ψ which enter Eqs. (32)–(34) can be easily calculated. From the defining Eq. (16b) one has:

$$\Phi^{(2)} = \frac{16\sqrt{2}\pi}{15}\left(\frac{\varrho_c}{\alpha_n\beta_n}\right)^{5/2n}\theta^{5/2}\,, \qquad (41a)$$

$$\Phi^{(4)} = \frac{32\sqrt{2}\pi}{105}\left(\frac{\varrho_c}{\alpha_n\beta_n}\right)^{7/2n}\theta^{7/2}\,. \qquad (41b)$$

From Eqs. (16b), (37) and (39) one gets:

$$\Psi^{(0)} = \left(8\sqrt{2}\pi\left|n-\frac{3}{2}\right|\alpha_n\right)^{1/2}\beta_{(2n+1)/4}\cdot\left(\frac{\varrho_c}{\alpha_n\beta_n}\right)^{(2n+1)/4n}\theta^{(2n+1)/4}\,. \qquad (42a)$$

From Eqs. (42a) and (15c) one obtains:

$$\Psi^{(2)} = \left(2\sqrt{2}\pi\left|n-\frac{3}{2}\right|\alpha_n\right)^{1/2}\frac{8}{2n+5} \cdot \beta_{(2n+1)/4}\left(\frac{\varrho_c}{\alpha_n\beta_n}\right)^{(2n+5)/4n}\theta^{(2n+5)/4}. \tag{42b}$$

Substitution of Eqs. (41)–(42) in Eqs. (32)–(34), and changing the variable of integration to $x=r/R=\zeta/\zeta_0$, where ζ_0 is the Lane-Emden radius of the polytrope, gives:

$$S = AR^3\int_0^1 \theta^{5/2}(x)\xi^2(x)x^2dx\,, \tag{43}$$

$$W_1 = AR\left(\frac{\varrho_c}{\alpha_n\beta_n}\right)^{1/n}\left\{\frac{6}{7}\int_0^1\theta^{7/2}\left(\frac{d\xi}{dx}\right)^2x^2dx + \int_0^1\left[\frac{12}{7}\theta^{7/2}+2x\theta^{5/2}\frac{d\theta}{dx}+\zeta_0^2x^2\theta^{n+5/2}\right]\xi^2(x)dx\right\}, \tag{44}$$

$$W_2 = CAR\left(\frac{\varrho_c}{\alpha_n\beta_n}\right)^{1/n}\zeta_0^2\int_0^1\theta^{n+5/2}\xi^2(x)x^2dx\,, \tag{45}$$

where the constants A and C are

$$A = \frac{64\sqrt{2}\pi^2}{15}\left(\frac{\varrho_c}{\alpha_n\beta_n}\right)^{5/2n}, \tag{46a}$$

$$C = \frac{120|n-\frac{3}{2}|}{(2n+5)^2}\frac{\beta^2_{(2n+1)/4}}{\beta_n}. \tag{46b}$$

Note that C is dimensionless and vanishes for $n=\frac{3}{2}$. The integrals in Eqs. (43)–(45) are also dimensionless. This leaves

$\dim(W_1+W_2)/S = (\varrho_c/\alpha_n\beta_n)^{1/n}/R^2 = 4\pi G\varrho_c\zeta_0^2$.

The factor $4\pi G\varrho_c$ is chosen to be the unit of the eigenvalues ω^2.

Numerical results

The eigenvalue equation $\omega^2=[W_1+\mathrm{sign}(F_E)W_2]/S$, for polytropes is solved by a Rayleigh-Ritz variational procedure. A brief description of the method along with a short account of the trial functions is given in Appendix A. The eigenvalues corresponding to from one to eight variational parameters, and for polytropes $n=1$, 1.5, 2, 2.5, 3, 3.5, 4, and 4.5 are given in Table 1. The eigenvalues are in units of $4\pi G\varrho_c$. The seventh and eigth eigenvalues in Table 1 indicate orders of magnitude only. They are, however, included to help to appraise the convergence of the variational results. The rapid convergence of the eigenvalues should be credited to the completeness of the set of the trial functions of Eq. (A.6) below. No negative eigenvalue, i.e., no instability is detected at this stage.

Polytropes of high index have high central condensations and are physically extended systems. In Table 1 as the polytropic index increases the eigenvalues decrease monotonically. We have found that both W_1 and W_2 integrals decrease to zero with n increasing to 5. The W_2-term, however, decreases much faster than the W_1-term. This behavior is shown in Table 2, where ω^2, W_1, and W_2 for the first radial modes are calculated for various polytropes. The W_2-term expresses the effects of the self-gravitation perturbations. For $0.5<n<1.5$ it contributes to the stability of the system. For $n=1.5$ it is zero, and for $n>1.5$ it decreases the eigenvalues.

By the theorem of Sect. II the polytropes $0.5<n<1.5$ are stable; for $\mathrm{sign}(F_E)=+1$ and $F|F_E|^{-1/2}=0$ at $E=0$. By the work of Dorémus et al. (1971) the polytropes $n>1.5$ are also stable, for

Table 1. Radial eigenvalues of polytropic stellar systems, using one to eight variational parameters. Eigenvalues are in units of $4\pi G\varrho_c$. N is the polytropic index. A number $a\times10^{\pm b}$ is written as a $\pm b$

0.3250+1								
0.2592+1	0.5291+1							
0.2438+1	0.4368+1	0.7722+1						N = 1.0
0.2415+1	0.4121+1	0.6492+1	0.1070+1					
0.2413+1	0.4077+1	0.6143+1	0.9104+1	0.1428+2				
0.2413+1	0.4073+1	0.6075+1	0.8622+1	0.1225+2	0.1846+2			
0.2413+1	0.4073+1	0.6067+1	0.8518+1	0.1160+2	0.1595+2	0.2325+2		
0.2413+1	0.4073+1	0.6067+1	0.8505+1	0.1145+2	0.1509+2	0.2021+2	0.2867+2	
0.5467								
0.4900	0.1662+1							
0.4875	0.1400+1	0.3219+1						N = 1.5
0.4875	0.1372+1	0.2641+1	0.5228+1					
0.4875	0.1371+1	0.2546+1	0.4250+1	0.7686+1				
0.4875	0.1371+1	0.2538+1	0.4034+1	0.6239+1	0.1060+2			
0.4875	0.1371+1	0.2537+1	0.4005+1	0.5847+1	0.8620+1	0.1396+2		
0.4875	0.1371+1	0.2537+1	0.4003+1	0.5774+1	0.7997+1	0.1140+2	0.1778+2	
0.1733								
0.1635	0.8243							
0.1633	0.6761	0.1784+1						N = 2.0
0.1633	0.6630	0.1371+1	0.3079+1					
0.1633	0.6626	0.1308+1	0.2307+1	0.4702+1				
0.1633	0.6626	0.1303+1	0.2140+1	0.3500+1	0.6647+1			
0.1633	0.6626	0.1303+1	0.2118+1	0.3173+1	0.4961+1	0.8913+1		
0.1633	0.6626	0.1303+1	0.2117+1	0.3109+1	0.4421+1	0.6695+1	0.1150+2	
0.1019								
0.9384-1	0.4479							
0.9360-1	0.3388	0.1008+1						N = 2.5
0.9360-1	0.3294	0.6982	0.1790+1					
0.9360-1	0.3291	0.6522	0.1201+1	0.2791+1				
0.9360-1	0.3291	0.6486	0.1077+1	0.1860+1	0.4008+1			
0.9360-1	0.3291	0.6485	0.1060+1	0.1616+1	0.2587+1	0.5437+1		
0.9360-1	0.3291	0.6485	0.1059+1	0.1566+1	0.2279+1	0.3685+1	0.7078+1	
0.7173-1								
0.5838-1	0.2390							
0.5779-1	0.1588	0.5310						N = 3.0
0.5779-1	0.1519	0.3216	0.9527					
0.5779-1	0.1517	0.2925	0.5574	0.1505+1				
0.5779-1	0.1517	0.2902	0.4820	0.8761	0.2187+1			
0.5779-1	0.1517	0.2901	0.4712	0.7267	0.1285+1	0.2997+1		
0.5779-1	0.1517	0.2901	0.4705	0.6959	0.1034+1	0.1788+1	0.3936+1	
0.4452-1								
0.2746-1	0.1197							
0.2656-1	0.6502-1	0.2477						N = 3.5
0.2655-1	0.6024-1	0.1258	0.4358					
0.2655-1	0.6001-1	0.1103	0.2155	0.6875				
0.2655-1	0.6001-1	0.1089	0.1789	0.3392	0.1005+1			
0.2655-1	0.6001-1	0.1088	0.1733	0.2688	0.5011	0.1388+1		
0.2655-1	0.6001-1	0.1088	0.1729	0.2541	0.3835	0.7048+0	0.1838+1	
0.2040-1								
0.7857-2	0.4950-1							
0.7366-2	0.1876-1	0.9342-1						N = 4.0
0.7353-2	0.1649-1	0.3573-1	0.1555					
0.7353-2	0.1634-1	0.2956-1	0.6022-1	0.2382				
0.7353-2	0.1634-1	0.2884-1	0.4725-1	0.9375-1	0.3435			
0.7353-2	0.1634-1	0.2880-1	0.4504-1	0.7044-1	0.1378	0.4726		
0.7353-2	0.1634-1	0.2880-1	0.4483-1	0.6528-1	0.1001	0.1936	0.6266	
0.4603-2								
0.8400-3	0.1096-1							
0.7734-3	0.2086-2	0.1964-1						N = 4.5
0.7711-3	0.1754-2	0.4085-2	0.3094-1					
0.7711-3	0.1728-2	0.3171-2	0.7002-2	0.4511-1				
0.7711-3	0.1728-2	0.3052-2	0.5109-2	0.1099-1	0.6257-1			
0.7711-3	0.1728-2	0.3046-2	0.4763-2	0.7666-2	0.1620-1	0.8347-1		
0.7711-3	0.1728-2	0.3045-2	0.4727-2	0.6902-2	0.1094-1	0.2278-1	0.1081	

their sufficient condition, $F_E<0$, is fulfilled. That Table 1 contains no negative eigenvalue for any of the polytropes is in agreement with these stability theorems.

Table 3 contains a sample of eigenvectors. Eigenvalues are displayed in lines marked by an asterisk. The column following an eigenvalue is its corresponding eigenvector. Computations are for eight variational parameters. To illustrate the use of the table we give an example. For $n=1.5$ the first eigenfunction corresponding to $\omega^2=0.4875$ is $\xi(x)=0.6063x+0.7985x^3+0.5507x^5+\ldots$, where $x=r/R$, and the numerical constants are taken from the first column of the data for $n=1.5$.

More important then ξ, however, are the density perturbations of Eqs. (36). Sample calculations are plotted in Figs. 1 and 2. In

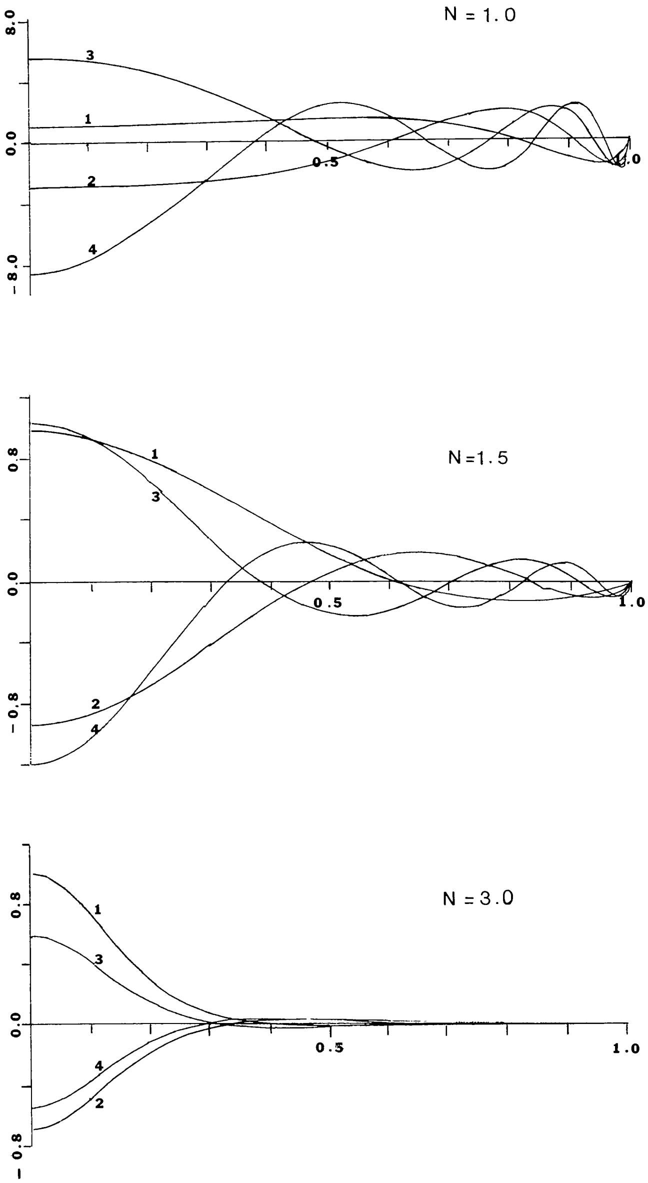

Fig. 1. Radial volume density waves of polytropes 1, 1.5, and 3 as functions of the radius, Numbers 1–4 on the curves indicate the mode order. The radius is normalized to 1. The amplitude of the first mode is normalized to 1

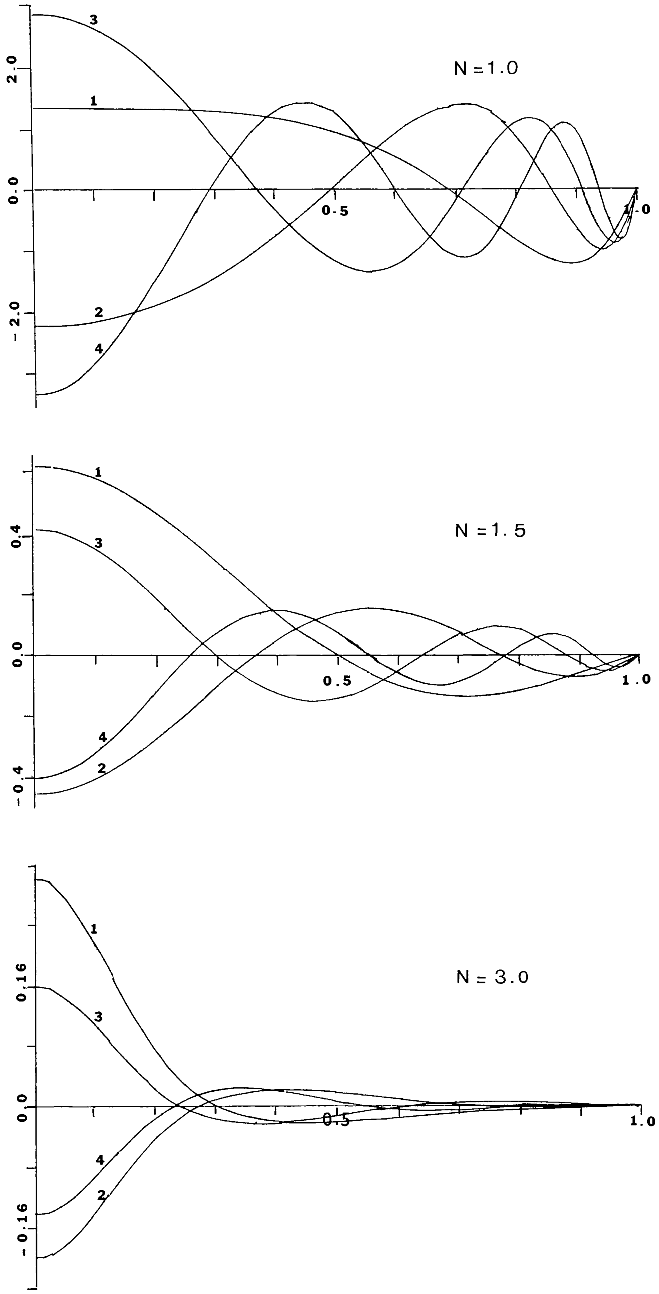

Fig. 2. Radial surface density waves of polytropes 1, 1.5, and 3. See the legend for Fig. 1 for further specifications

Table 2. Contributions of W_1 and W_2 to the first eigenvalue. The W_2-term arises from perturbation in the self-gravitation and is much smaller than the W_1-term. A number $a\times 10^{\pm b}$ is written as $a\pm b$

N	ω^2	W_1	sign$(F_E)W_2$
1.0	0.2413+1	0.1154+1	0.1259+1
1.5	0.4875	0.4875	0.0
2.0	0.1633	0.3337	-0.1703
2.5	0.9360-1	0.2022	-0.1086
3.0	0.5779-1	0.9465-1	-0.3686-1
3.5	0.2655-1	0.3230-1	-0.5749-2
4.0	0.7353-2	0.7713-2	-0.3597-3
4.5	0.7711-3	0.7763-3	-0.5140-5

polytrope 1 both volume and surface densities extend outward and cover considerable fractions of the radius. In the centrally condensed polytrope 3, both perturbations are very much localized near the center and have negligible amplitudes after their first nodes. All density perturbations have a node at the surface and an antinode at the center. In addition, the first mode has one more node somewhere in between, the second two more nodes, and so on.

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Table 3. Radial eigenvalues and eigenvectors of some polytropes calculated with eight variational parameters. Eigenvalues are displayed in lines marked by an asterisk. The column below an eigenvalue is the corresponding eigenvector. N is the polytropic index. A number $a\times 10^{\pm b}$ is written as $a\pm b$

N = 1.0							
*0.2413+1	0.4073+1	0.6067+1	0.8505+1	0.1145+2	0.1509+2	0.2021+2	0.2867+2
0.1383	-0.5408	0.1229+1	-0.2251+1	0.3013+1	0.6109+1	0.5744+1	-0.2057+2
0.5921	-0.1198+1	0.5549	0.9354+1	-0.5339+1	-0.9966+2	-0.2358+2	0.5469+3
0.1089+1	-0.1529+1	-0.1250+2	-0.3768+2	-0.2228+3	0.7945+3	-0.1052+4	-0.6829+4
0.1953+1	0.7531+1	0.4466+2	0.3306+3	0.1596+4	-0.4581+4	0.1040+5	0.3385+5
-0.4287	-0.1898+2	-0.1922+3	-0.1237+4	-0.5450+4	0.1645+5	-0.3850+5	-0.8890+5
0.4687+1	0.4977+2	0.3832+3	0.2369+4	0.1065+5	-0.3192+5	0.6838+5	0.1271+6
-0.3420+1	-0.4565+2	-0.3947+3	-0.2501+4	-0.1088+5	0.3037+5	-0.5852+5	-0.9338+5
0.2396+1	0.2653+2	0.1987+3	0.1116+4	0.4371+4	-0.1109+5	0.1937+5	0.2757+5
N = 1.5							
*0.4875	0.1371+1	0.2537+1	0.4003+1	0.5774+1	0.7997+1	0.1140+2	0.1778+2
0.6063	-0.9567	0.1451+1	-0.2084+1	0.2615+1	0.4895+1	0.4464+1	-0.1823+2
0.7985	0.5277-1	-0.2708+1	0.9538+1	-0.9657+1	-0.8332+2	0.5251+0	0.5891+3
0.5507	0.2278+1	-0.6823+1	-0.6842+1	-0.1493+3	0.7111+3	-0.1248+4	-0.6360+4
0.2465	0.3052+1	-0.1253+1	0.9584+2	0.1170+4	-0.4432+4	0.1138+5	0.3213+5
0.1455	0.3975+1	-0.2861+1	-0.4567+3	-0.4125+4	0.1677+5	-0.4149+5	-0.8571+5
-0.1040	-0.3690	0.2922+2	0.8843+3	0.8497+4	-0.3359+5	0.7360+5	0.1242+6
0.1153	0.2957+1	-0.1597+2	-0.1026+4	-0.9233+4	0.3267+5	-0.6316+5	-0.9238+5
-0.5882-1	-0.7223	0.2477+2	0.5528+3	0.3926+4	-0.1213+5	0.2099+5	0.2756+5
N = 2.0							
*0.1633	0.6626	0.1303	0.2117+1	0.3109+1	0.4421+1	0.6695+1	0.1150+2
0.7924	-0.8900	0.1207+1	0.1607+1	0.1881+1	0.3815+1	-0.3213+1	-0.1600+2
0.5316	0.3850	-0.2225+1	-0.6584+1	-0.2127+1	-0.6831+2	-0.3206+2	0.5291+3
0.2488	0.2466+1	-0.6646+1	0.2276+1	-0.1771+3	0.6999+3	0.1548+4	-0.5813+4
0.1142	0.3663+1	-0.4330+1	-0.9547+2	0.1300+4	-0.5015+4	-0.1281+5	0.2978+5
-0.2726-1	0.3295+1	0.1377+1	0.4432+3	-0.4857+4	0.1983+5	0.4522+5	-0.8037+5
0.8270-2	0.1305+1	0.2055+2	-0.8697+3	0.1053+5	-0.4000+5	-0.7897+5	0.1177+6
-0.7756-2	0.6822	0.2564+1	0.1094+4	-0.1169+5	0.3874+5	0.6717+5	-0.8823+5
0.1868-2	-0.2093	0.1697+2	-0.6299+3	0.4986+4	-0.1429+5	-0.2220+5	0.2652+5
N = 3.0							
*0.5779-1	0.1517	0.2901	0.4705	0.6959	0.1034+1	0.1788+1	0.3936+1
0.5675	-0.6355	0.7431	-0.8873	-0.9165	0.2125+1	0.1518+1	-0.1150+2
0.1039+1	-0.8326	0.4183	0.1169+1	-0.7193+1	-0.4108+2	0.6883+2	0.4077+3
0.9912	0.1516+1	-0.6331+1	-0.1809+1	0.1980+3	0.6321+3	-0.1866+4	-0.4695+4
0.6733	0.6264+1	-0.1365+2	0.1536+3	-0.1448+4	-0.5687+4	0.1422+5	0.2491+5
-0.9956	0.6016+1	-0.1156+2	-0.5967+3	0.6195+4	0.2402+5	-0.4872+5	-0.6913+5
0.1421+1	0.1059+1	0.4974+2	0.1222+4	-0.1466+5	-0.4909+5	0.8382+5	0.1035+6
-0.1004+1	0.1256+1	-0.1311+2	-0.1688+4	0.1677+5	0.4741+5	-0.7068+5	-0.7912+5
0.2929	-0.5626-1	0.3313+2	0.9917+3	-0.7168+4	-0.1736+5	0.2323+5	0.2417+5

Appendix A: computational technique

a) The Rayleigh-Ritz procedure

The eigenvalue equations encountered in Sect. V are of the form:

$$\mathscr{W}\xi^{\nu}=\lambda^{\nu}\Phi\xi^{\nu} \tag{A.1a}$$

or

$$\lambda^{\nu}=\int\xi^{\nu*}\mathscr{W}\xi^{\nu}d\boldsymbol{x}\Big/\int\xi^{\nu*}\Phi\xi^{\nu}d\boldsymbol{x}\,. \tag{A.1b}$$

The eigenfunction ξ^{ν} is either a general vector field, as in Eqs. (27)–(31) or a radial vector field, as in Eqs. (32)–(36). In addition, W is self adjoint and Φ is positive definite. Let $\{\zeta^{\lambda}, \lambda=1,2,\ldots\}$ be a set of complete trial functions which satisfy the same boundary conditions as $\{\xi^{\nu}, \nu=1,2,\ldots\}$. Expand ξ^{ν} in terms of $\{\zeta^{\lambda}\}$:

$$\xi^{\nu}=\sum_{\lambda}\zeta^{\lambda}Z^{\lambda\nu} \tag{A.2}$$

where $Z^{\lambda\nu}$ are linear expansion coefficients and will be used as variational parameters. Substitute Eq. (A.2) in Eq. (A.1a), pre-multiply by ζ^{μ} and integrate over the x-space. One gets:

$$W^{\mu\lambda}Z^{\lambda\nu}=S^{\mu\lambda}Z^{\lambda\nu}\lambda^{\nu}, \tag{A.3}$$

where the matrix elements $W^{\mu\lambda}$ and $S^{\mu\lambda}$ are:

$$W^{\mu\lambda}=\int\zeta^{\mu*}\mathscr{W}\zeta^{\lambda}d\boldsymbol{x},$$

$$S^{\mu\lambda}=\int\zeta^{\mu*}\Phi\zeta^{\lambda}d\boldsymbol{x}.$$

Let $\Lambda=[\lambda^{\nu}]_{\rm diagonal}$ be the diagonal matrix of the eigenvalues, and W, S, Z be the matrices with elements $W^{\mu\lambda}$, $S^{\mu\lambda}$, and $Z^{\mu\lambda}$ respectively, Eq. (A.3) for all eigenvectors becomes:

$$WZ=SZ\Lambda\,. \tag{A.5a}$$

Similarly, the matrix form of Eq. (A.1b) becomes:

$$\Lambda=Z^{\dagger}WZ/Z^{\dagger}SZ\,. \tag{A.5b}$$

The matrices in Eqs. (A.5) are, in general, infinite matrices. The Rayleigh-Ritz procedure consists of approximating them by finite square matrices and solving Eqs. (A.5) for the approximate Λ and Z.

b) The trial functions

The basis $\{\zeta^{\lambda}\}$ should satisfy the same boundary condition as $\{\xi^{\nu}\}$ and should be complete. In a remark after Eq. (2a), it was noted that $\xi(\boldsymbol{x})$ is an indicator of the macroscopic motion of the perturbed system. Equation of continuity then requires div ξ to be finite at the origin and at the boundary. This statement can be proved rigorously from Eqs. (3a), (5b), and (19b). We will elaborate on this point in a forthcoming paper where we analyze non-radial perturbations. With these considerations in mind, for radial perturbations we have chosen:

$$\{\zeta^{\mu}\}=\{x^{2\mu-1}, \mu=1,2,\ldots\}, \tag{A.6}$$

where $x=r/R$ is the fractional variable. Dixit et al. (1980) show that the proposed set is complete in the range $0<x<1$, and any *well-defined* function can be expanded in terms of it.

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