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ASTRONOMY AND ASTROPHYSICS

Linear oscillations of isotropic stellar systems

II. Radial modes of energy-truncated models*

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Summary. The eigenvalue problem for the first order radial perturbation of four energy-truncated distributions is solved. Wooley's and Michie-King's models are among the examples analyzed. All eigenvalues of all models with finite masses and radii are real. There is no indication of instability in dynamical time scales.

Key words: stellar systems – normal modes-Antonov's equation – Liouville's equation-density waves

1. Introduction

Linear perturbations of the collisionless Boltzmann-Liouville equation are governed by Antonov's equation. This is an eigenvalue equation in the six dimensional phase space. In Paper I of this series (Sobouti, 1984) it was proposed to Fourier-expand the phase space density perturbations in the velocity space. This reduced Antonov's equation into a series of equations involving functions of space coordinates only. Solutions in successive approximations became feasible. In the present paper the method is applied to the radial perturbations of the energy-truncated models of Woolley, Michie-King, and to other models in the sequence. The first order radial modes are calculated.

The equilibrium structure of four energy truncated models is summarized in Section 2. The first order perturbation equation is laid out in Sect. 3. The numerical results and a discussion constitute the object of Sects. 4 and 5. A proof of stability against first order perturbations is given in the appendix.

2. The equilibrium structure of the energy-truncated models

Self gravitating isothermal gas spheres with Maxwellian distributions are rejected as model stellar systems. They allow velocities greater than the escape velocity, have infinite masses and extend to infinity (see Chandrasekhar, 1957, pp. 155–168). As a plausible but arbitrary model, Woolley (1954) adopted a truncated Maxwellian distribution,

$$F_1(E)=\left(\frac{\alpha}{2\pi}\right)^{3/2} C_1 e^{-\alpha E} H(-E), \tag{1a}$$

where C_1 and α are constants, $E=\frac{1}{2}v^2+U(r)$ is the energy integral, and U is the gravitational potential. The step function is $H(-E)=1$ if $E<0$, $=\frac{1}{2}$ if $E=0$, and $=0$ if $E>0$. Michie (1963) proposed the distribution

$$F_2(E)=\left(\frac{\alpha}{2\pi}\right)^{3/2} C_2(e^{-\alpha E}-1)H(-E). \tag{1b}$$

This distribution in addition to being an exact solution of the Boltzmann-Liouville equation, is an approximate solution of the Fokker-Planck equation as well. The latter takes stellar encounters into account. Over time spans of relaxation time scales, it is a better dynamical equation to use (Chandrasekhar, 1960). King (1966) employed Michie's distribution and calculated the space, and the projected-on-the sky densities of some globular cluster models. Evidently, King's models fit reasonably well some of the observations. Katz (1980) reports that the distribution

$$F_3(E)=\left(\frac{\alpha}{2\pi}\right)^{3/2} C_3(e^{-\alpha E}-1+\alpha E)H(-E) \tag{1c}$$

fits the open isolated cluster models of Spitzer and Thuan rather well. To this list we add

$$F_4=\left(\frac{\alpha}{2\pi}\right)^{3/2} C_4\left(e^{-\alpha E}-1+\alpha E-\frac{1}{2}\alpha^2E^2\right)H(-E). \tag{1d}$$

This distribution could be handled along the same formalism as the other three. We shall show below that F_1, F_2, F_3, and F_4 are the only ones in the list of the truncated Maxwellian distributions that could yield finite models.

The potential $U(r)$ implicit in Eqs. (1) will be chosen zero at the boundary of the system. This choice implies that the velocity $v_e(r)=(-2U)^{1/2}$ is the velocity of escape from r to the boundary rather than to the infinity. King argues that for star clusters in the gravitational field of a galaxy this is the proper parameter to use. For, once a star reaches the boundary of the cluster it will be lost in the tidal force field of the galaxy.

The mass density is $\varrho_j=4\pi\int_0^{v_e} F_j(E)v^2dv$. Elementary manipulations of Eqs. (1) gives

$$\varrho_j(\theta)=C_j\mathscr{E}_j(\theta), \quad j=1,2,3,4, \tag{2}$$

where

$$\theta=-\alpha U \tag{2a}$$

and

$$\mathscr{E}_j(\theta)=\frac{2^{j-1}}{1.3\ldots(2J-1)}\frac{2}{\sqrt{\pi}}e^{\theta}\int_0^{\theta} e^{-t}t^{j-\frac{1}{2}}dt \tag{3}$$

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(see also Katz, 1980). The following recursion relations exists for the $\mathscr{E}$-functions:

$$\mathscr{E}_j=\mathscr{E}_{j-1}-\frac{2^{j-1}}{1.3\ldots(2j-1)}\frac{2}{\sqrt{\pi}}\theta^{j-\frac{1}{2}} \quad (3a)$$

$$\mathscr{E}_0=e^\theta \operatorname{Erf}(\sqrt{\theta})=e^\theta\frac{2}{\sqrt{\pi}}\int_0^{\sqrt{\theta}} e^{-u^2}du. \quad (3b)$$

Also

$$\frac{d\mathscr{E}_j}{d\theta}=\mathscr{E}_{j-1}(\theta) \quad (3c)$$

and

$$\mathscr{E}_j(\theta)\rightarrow\theta^{j+\frac{1}{2}} \quad \text{as} \quad \theta\rightarrow 0. \quad (3d)$$

The Poisson equation

Let θ_c denote the central value of the dimensionless potential θ. The density at the center is $\varrho_c=C_j\mathscr{E}_j(\theta_c)$. Let us define a dimensionless radius $\zeta=(4\pi G\alpha\varrho_c)^{1/2}r$, where G is the gravitational constant: the Poisson equation, $\nabla^2U=4\pi G\varrho$, may now be expressed in the dimensionless form

$$\frac{d^2\theta}{d\zeta^2}+\frac{2}{\zeta}\frac{d\theta}{d\zeta}=-\mathscr{E}_j(\theta)/\mathscr{E}_j(\theta_c). \quad (4)$$

This equation is to be solved with the conditions

$$\theta=\theta_c, \quad \text{and} \quad \frac{d\theta}{d\zeta}=0 \quad \text{at} \quad \zeta=0. \quad (4a)$$

Evidently, Eqs. (4) do not admit a homology transformation of the type that polytropes or isothermal gas spheres do (see Chandrasekhar, 1957, pp. 101–106, 158–160, for homology theorems). This means that θ_c in Eqs. (4) cannot be transformed, and that for each model of a given θ_c a numerical integration of Eqs. (4) is necessary. Considering the asymptotic behavior of Eq. (3d), however, Eqs. (4) in the limit $\theta_c\rightarrow 0$ reduce to the polytropic equation of index $j+\frac{1}{2}$. Since a polytropic index cannot be larger than five; one has to be contended with the values $j=1,2,3,4$ only. Since the corresponding asymptotic polytropes are finite configurations all four models with low enough θ_c should have finite masses and radii. With increasing θ_c, however, the masses and the radii grow rapidly and become infinite at some finite limiting values of θ_c. The higher j, the faster the models diverge. A choice of θ_c or its dimensional equivalent, $U_c=-\alpha\theta_c$, sets the upper bound for the velocity in the cluster. This parameter will be said to indicate the *degree* of truncation. The subscript $j=1, 2, 3, 4$ on F's will be refered to as the *order* of truncation.

Table 1 lists some of the characteristics of the models. The unit of length is

$$R_c=(4\pi G\alpha\varrho_c)^{-1/2}. \quad (5a)$$

The values of θ_c and the surface radius $\zeta_0=R/R_c$ are given in columns 1 and 2, respectively. Column 3 lists the inflection point in ϱ. This is the radius at which the density begins to level off and could reasonably be identified as some sort of core-halo boundary. It is significant to note that, while the halo radius, ζ_0, ranges from 2 to 2000, the inflection radius, ζ_{inf}, stays between 1 and 4. Therefore, one can safely use R_c as an indicator of the physical radius of the core.

The virial theorem can be used to define a characteristic dimension for the cluster. Thus,

$$R_{\text{vir}}=GM^2/2|\textit{grav. pot. energy}|. \quad (5b)$$

It has been suggested that R_{vir} is a core radius (see, for example, King, 1967). The quantity $\zeta_{\text{vir}}=R_{\text{vir}}/R_c$ is given in column 4. In models with intermediate and with moderately high central densities, ζ_{vir} is of the same order of magnitude as ζ_0. Obviously it cannot be a good indicator of a core radius. Only in extremely high central densities does it drop appreciably below the surface radius.

To indicate the degree of the central concentrations (density contrast), $\varrho(\zeta_0/2)/\varrho_c$ is given in column 5. The relative density at inflection point, $\varrho_{\text{inf}}/\varrho_c$ is displayed in column 6.

The mass $M(r)=4\pi\int_0^r\varrho r^2dr$, is readily obtained by substituting for ϱ from the Poisson Eq. (4), and integrating by parts. Thus,

$$M(\zeta)=-4\pi R_c^3\varrho_c\zeta^2\frac{d\theta}{d\zeta}. \quad (6)$$

The dimensionless total mass, $M(\zeta_0)=-\zeta_0^2(d\theta/d\zeta)_0$, is given in column 7. The ratios $M(\zeta_0/2)/M(\zeta_0)$ and $M(\zeta_{\text{inf}})/M(\zeta_0)$ are given in columns 8 and 9, respectively.

3. The first order eigenvalue equation for radial perturbations

A scheme of expansion of the linearized Boltzmann-Liouville equation in phase space into a series of equations in the configuration space is developed in Paper I. The first order equation in the series, for radial perturbations, assumes that the antisymmetric perturbations of the phase space density are of the form $f(\boldsymbol{x},\boldsymbol{v},t)=\xi(r)v_r\exp(i\omega t)$, where v_r is the radial component of $\boldsymbol{v}$, and $\xi(r)$ is a radial vector field to be determined. This perturbation could be considered as a variational ansatz in the velocity. The eigenvalues corresponding to it could only be larger than the exact ones.

The equilibrium distributions of Eqs. (1) all have $dF/dE<0$. For such distributions, the equation governing $\xi(r)$ is [Paper 1, Eqs, (35)]:

$$\omega^2\Phi^{(2)}\xi=\mathscr{W}_1\xi-\mathscr{W}_2\xi, \quad (7)$$

where

$$\mathscr{W}_1\xi=-\frac{3}{r^2}\frac{d}{dr}\left(r^2\Phi^{(4)}\frac{d\xi}{dr}\right)+\left(\frac{6}{r^2}\Phi^{(4)}+\frac{2}{r}\frac{d\Phi^{(4)}}{dr}+4\pi G\varrho\Phi^{(2)}\right)\xi, \quad (7a)$$

$$\mathscr{W}_2\xi=16\pi^2G[\Psi^{(2)}]^2r^2\xi. \quad (7b)$$

The functions $\Phi^{(2)}$, $\Phi^{(4)}$, and $\Psi^{(2)}$ are equilibrium parameters, and are defined as follows (Paper I, Eqs. 16):

$$\Phi^{(n)}=\frac{4\pi}{1.3\ldots(n+3)}(-2U)^{(n+3)/2}, \quad n=0,2,4,\ldots, \quad (8a)$$

$$\Psi^{(n)}=\frac{4\pi}{1.3\ldots(n+1)}\int_0^{\sqrt{-2U}}|dF/dE|^{1/2}v^{n+2}dv, \quad n=0,2,4,\ldots. \quad (8b)$$

The operators $\mathscr{W}_1$ and $\mathscr{W}_2$ are Hermitian and positive. The eigenvalues, ω^2, are real and are obtainable from the variational expression.

$$\omega^2=(W_1-W_2)/S, \quad (9)$$

Table 1. Equilibrium parameter of energy-truncated distributions

θ_c	ζ_o	ζ_{inf}	ζ_{vir}	$\rho(\zeta_{o/2})$	$\rho(\zeta_{inf})$	M_{tot}	$M(\zeta_{o/2})$	$M(\zeta_{inf})$
j = 1 (Woolley's model):								
0.001	0.365-1	0.147-1	0.512-1	0.420	0.576	0.271-5	0.460	0.283
1.0	0.405+1	0.120+1	0.495+1	0.297	0.637	0.263+1	0.525	0.167
2.0	0.658+1	0.141+1	0.683+1	0.180	0.674	0.725+1	0.600	0.101
4.0	0.141+2	0.153+1	0.982+1	0.374-1	0.694	0.203+2	0.744	0.469-1
6.0	0.338+2	0.156+1	0.148+2	0.380-2	0.695	0.416+2	0.827	0.243-1
8.0	0.102+3	0.164+1	0.330+2	0.298-3	0.674	0.957+2	0.329	0.120-1
10.0	0.314+3	0.188+1	0.114+3	0.373-4	0.604	0.287+3	0.789	0.570-2
12.0	0.838+3	0.335+1	0.351+3	0.612-5	0.289	0.851+3	0.780	0.679-2
14.0	0.218+4	0.437+1	0.906+3	0.890-6	0.174	0.229+4	0.792	0.400-2
j = 2 (Michie = King's model):								
0.001	0.536-1	0.106-1	0.383-1	0.881-1	0.639	0.219-5	0.777	0.139
1.0	0.592+1	0.972	0.378+1	0.559-1	0.657	0.216+1	0.816	0.110
2.0	0.960+1	0.125+1	0.534+1	0.305-1	0.674	0.606+1	0.854	0.843-1
4.0	0.208+2	0.149+1	0.792+1	0.508-2	0.683	0.174+2	0.919	0.508-1
6.0	0.540+2	0.151+1	0.121+2	0.338-3	0.703	0.363+2	0.954	0.256-1
8.5	0.205+3	0.164+1	0.279+2	0.126-4	0.673	0.857+2	0.960	0.134-1
10.0	0.673+3	0.269+1	0.105+3	0.151-5	0.408	0.270+3	0.945	0.137-1
12.0	0.164+4	0.329+1	0.316+3	0.329-6	0.299	0.794+3	0.941	0.702-2
j = 3:								
0.001	0.954-1	0.887-2	0.318-1	0.336-2	0.657	0.189-5	0.965	0.926-1
1.0	0.108+2	0.843	0.319+1	0.178-2	0.661	0.189+1	0.973	0.827-1
2.0	0.181+2	0.112+1	0.456+1	0.782-3	0.673	0.535+1	0.981	0.694-1
3.0	0.280+2	0.129+1	0.573+1	0.269-3	0.683	0.996+1	0.987	0.566-1
4.0	0.447+2	0.143+1	0.697+1	0.661-4	0.676	0.158+2	0.992	0.490-1
5.0	0.808+2	0.145+1	0.860+1	0.984-5	0.695	0.233+2	0.995	0.353-1
6.0	0.187+3	0.150+1	0.113+2	0.644-6	0.696	0.343+2	0.997	0.263-1
7.0	0.732+3	0.293+1	0.178+2	0.920-8	0.349	0.541+2	0.999	0.799-1
j = 4:								
0.001	0.318	0.764-2	0.289-1	0.0	0.667	0.174-5	0.99936	0.671-1
0.5	0.257+2	0.566	0.206+1	0.106-5	0.638	0.617	0.99950	0.747-1
1.0	0.428+2	0.770	0.295+1	0.474-6	0.653	0.175+1	0.99966	0.672-1
1.5	0.643+2	0.900	0.366+1	0.176-6	0.670	0.325+1	0.99976	0.589-1
2.0	0.971+2	0.116+1	0.431+1	0.491-7	0.604	0.505+1	0.99985	0.771-1
2.5	0.157+3	0.126+1	0.493+1	0.875-8	0.617	0.715+1	0.99992	0.690-1
3.0	0.319+3	0.128+1	0.560+1	0.523-9	0.649	0.959+1	0.99997	0.554-1
3.5	0.172+4	0.343+1	0.429+1	0.37-12	0.147	0.125+2	1.00000	0.337

Table 2. First order radial eigenvalues of energy-truncated distributions

θ_c	ω_1^2	ω_2^2	ω_3^2	ω_4^2	ω_5^2
j = 1 (Woolley's model):					
0.001	0.651-3	0.183-1	0.339-2	0.534-2	0.770-2
1.0	0.547+1	0.158+2	0.297+2	0.472+2	0.684+2
2.0	0.927+1	0.268+2	0.507+2	0.811+2	0.118+3
4.0	0.136+2	0.359+2	0.676+2	0.108+3	0.158+3
6.0	0.137+2	0.331+2	0.605+2	0.961+2	0.140+3
8.0	0.109+2	0.260+2	0.471+2	0.745+2	0.108+3
10.0	0.102+2	0.247+2	0.453+2	0.717+2	0.104+3
12.0	0.110+2	0.271+2	0.497+2	0.790+2	0.115+3
14.0	0.115+2	0.280+2	0.514+2	0.816+2	0.119+3
j = 2 (Michie - King's model):					
0.001	0.268-3	0.944-3	0.186-2	0.304-2	0.447-2
1.0	0.268+1	0.855+1	0.167+2	0.273+2	0.402+2
2.0	0.530+1	0.152+2	0.294+2	0.477+2	0.703+2
4.0	0.914+1	0.220+2	0.408+2	0.653+2	0.956+2
6.0	0.856+1	0.194+2	0.348+2	0.548+2	0.793+2
8.0	0.551+1	0.125+2	0.223+2	0.349+2	0.503+2
10.0	0.507+1	0.116+2	0.209+2	0.329+2	0.476+2
12.0	0.597+1	0.137+2	0.247+2	0.391+2	0.567+2
J = 3:					
0.001	0.242-3	0.547-3	0.991-3	0.157-2	0.229-2
1.0	0.221+1	0.495+1	0.888+1	0.140+2	0.204+2
2.0	0.389+1	0.866+1	0.154+2	0.241+2	0.349+2
3.0	0.483+1	0.108+2	0.190+2	0.296+2	0.427+2
4.0	0.490+1	0.109+2	0.193+2	0.300+2	0.430+2
5.0	0.408+1	0.913+1	0.161+2	0.250+2	0.358+2
6.0	0.261+2	0.586+1	0.103+2	0.160+2	0.230+2
7.0	0.106+1	0.238+1	0.421+1	0.653+1	0.936+1
J = 4:					
0.001	0.781-4	0.175-3	0.309-3	0.479-3	0.686-3
0.5	0.344	0.771+	0.136+1	0.211+1	0.302+1
1.0	0.589	0.132+1	0.233+1	0.362+1	0.518+1
1.5	0.726	0.163+1	0.288+1	0.446+1	0.639+1
2.0	0.749	0.168+1	0.297+1	0.461+1	0.660+1
2.5	0.657	0.148+1	0.260+1	0.404+1	0.579+1
3.0	0.434	0.976	0.172+1	0.268+1	0.383+1
3.5	0.105	0.236	0.416	0.646	0.926

where

$$S=4\pi\int_0^R \Phi^{(2)}\xi^2 r^2 dr>0\,, \tag{9a}$$

$$W_1=4\pi\left\{3\int_0^R \Phi^{(4)}\left(\frac{d\xi}{dr}\right)^2 r^2 dr + 2\int_0^R\left(3\Phi^{(4)}+r\frac{d\Phi^{(4)}}{dr}+2\pi G\varrho\Phi^{(2)}r^2\right)\xi^2 dr\right\}\geq 0\,, \tag{9b}$$

$$W_2=16\pi^2 G\int_0^R [\Psi^{(2)}]^2\xi^2 r^2 dr\geq 0\,. \tag{9c}$$

Reduction to dimensionless forms

From the defining Eqs. (8a) and (2a) one readily has

$$\Phi_j^{(2)}=\frac{4\pi}{15}\left(\frac{2}{\alpha}\right)^{5/2}\theta_j^{5/2}\,,\qquad \Phi_j^{(4)}=\frac{4\pi}{105}\left(\frac{2}{\alpha}\right)^{7/2}\theta_j^{7/2}\,, \tag{10}$$

where θ_j is an integral of Eqs. (4) for a given θ_c. The expression for $\Psi_j^{(2)}$ is somewhat involved. From Eqs. (8b), (1), and (3), however, one obtains

$$\Psi_j^{(2)}=4\pi^{3/4}C_j^{1/2}\left(\frac{2}{\alpha}\right)^{5/4}\eta_j(\theta_j)\,,\qquad j=1,2,3,4\,, \tag{11}$$

where

$$\eta_1(\theta_1)=\mathscr{E}_1(\theta_1/2)\,, \tag{11a}$$

$$\eta_2(\theta_2)=\mathscr{E}_1(\theta_2/2)\,,\quad \textit{note subscript “1” in } \mathscr{E}_1\,, \tag{11b}$$

$$\eta_3(\theta_3)=\frac{1}{3\sqrt{2\pi}}\int_0^{\theta_3}(e^{\theta_3-u}-1)^{1/2}u^{3/2}du\,, \tag{11c}$$

$$\eta_4(\theta_4)=\frac{1}{3\sqrt{2\pi}}\int_0^{\theta_4}[e^{\theta_4-u}-1-(\theta_4-u)]^{1/2}u^{3/2}du\,. \tag{11d}$$

In reducing the integrals S, W_1, and W_2, the fractional radius $x=r/R=\zeta/\zeta_0$ will be used. From Eqs. (9)–(12) one gets

$$S=\frac{16\pi^2}{15}R^3\left(\frac{2}{\alpha}\right)^5\int_0^1\theta_j^{5/2}\xi^2x^2dx\,, \tag{12a}$$

$$W_1=\frac{8\pi^2}{15}R\left(\frac{2}{\alpha}\right)^{7/2}\int_0^1\theta_j^{5/2}\left\{\frac{6}{7}\theta_j\left(\frac{d\xi}{dx}\right)^2x^2 + \left[\frac{12}{7}\theta_j+2\frac{d\theta_j}{dx}+\zeta_0^2x^2\mathscr{E}_j(\theta_j)/\mathscr{E}_j(\theta_c)\right]\xi^2\right\}dx\,, \tag{12b}$$

$$W_2=\frac{8\pi^2}{15}R\left(\frac{2}{\alpha}\right)^{7/2}[60\sqrt{\pi}\zeta_0^2/\mathscr{E}_j(\theta_c)]\int_0^1\eta_j^2(x)\xi^2x^2dx\,. \tag{12c}$$

The integrals in Eqs. (12) are dimensionless. The factor in the square bracket in Eq. (12c) is also dimensionless. This leaves $\dim(W_1-W_2)/S=1/\alpha R^2=4\pi G\varrho_c/\zeta_0^2$. In the numerical results of Table 2, this factor is chosen as the unit of the eigenvalues ω^2. The quantity $\zeta_0/\sqrt{4\pi G\varrho_c}$ is the crossing or dynamical time scale of the cluster.

4. Numerical results

The eigenvalue problem of Eq. (9), in which S, W_1, and W_2 are given by Eqs. (12), is solved by a Rayleigh-Ritz variational method. A description of the method in the context of astronomical problems is given by Silverman and Sobouti (1978) and also in the appendix to Paper I. The trial functions were $\xi_n(x) = \sum_{i=0}^{n} a_i x^{2i+1}$. Ten variational parameters, a_i, $i = 1, \ldots, 10$, were used in the computations.

A sample list of the eigenvalues is given in Table 2. The models with $\theta_c = 0.001$ are essentially the asymptotic polytropes 1.5, 2.5, 3.5, and 4.5, corresponding to $j = 1$, 2, 3, and 4, respectively. These asymptots are worked out in Paper I, and were of considerable help in the present work in checking and correcting the computer programs. All figures in the Table are significant.

Each eigenvalue, ω^2, is proportional to the difference $W_1 - W_2$, in which W_2 is due to perturbations in self gravitation, and has a destabilizing effect. In the appendix we show that $W_1 > W_2$ for all eigenvalues. So that all ω's are real and the truncated distributions are stable against the radial perturbations of the type considered here. Numerical integrations, however, reveal much more. In Figs. 1–4, ω^2, W_1/S, and W_2/S for the first mode are plotted against θ_c. The following features are noteworthy:

(a) In the first and second order truncation, W_2 is an order of magnitude less than W_1. In the third order the effect is much pronounced. In the fourth order this smallness amounts to two and three order of magnitude.

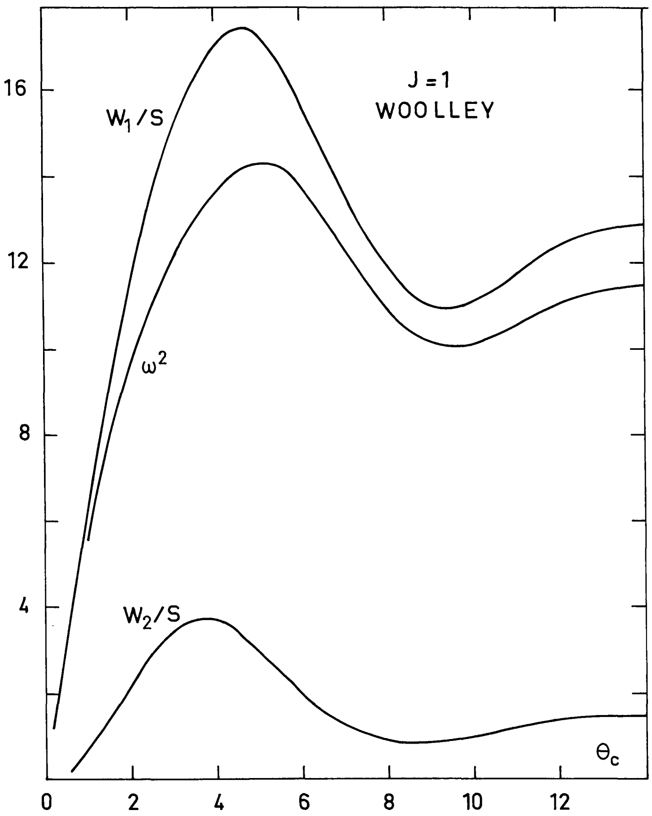

Fig. 1. Plots of ω^2, W_1/S, and W_2/S for the first radial mode of $j = 1$ models. The abscissa is the dimensionless central potential, θ_c. The unit on the ordinates is $4\pi G\varrho_c/\zeta_0^2$

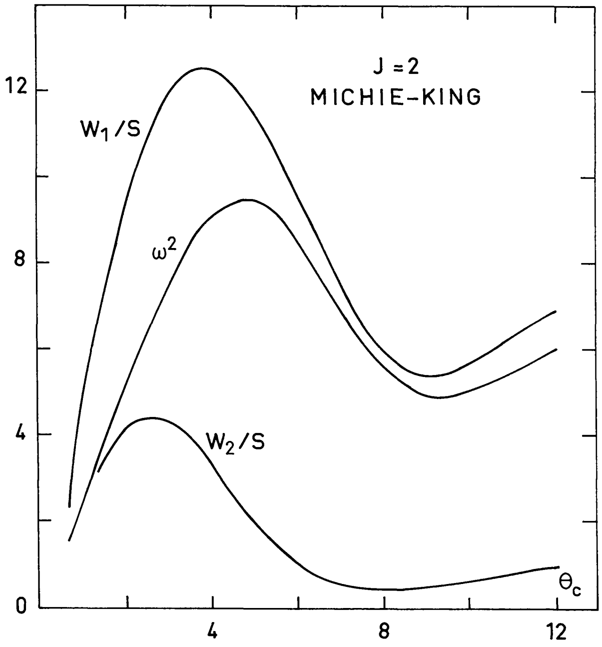

Fig. 2. Same as in Fig. 1 for $j = 2$ models

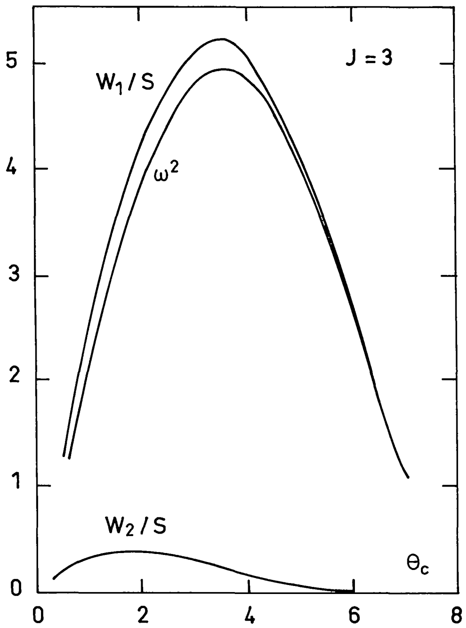

Fig. 3. Same as in Fig. 1 for $j = 3$ models

(b) With increasing mode number, W_2 becomes far less significant than W_1.

(c) With increasing θ_c (or equivalently, the central concentration) again W_2 falls rapidly behind W_1 and loses its role.

(d) In Figs. 1 and 2, W_1 and W_2 each display one maximum and one minimum. In Figs. 3 and 4 the minima disappear. The extrema for W_2 occur before those for W_1.

It should be noted that the choice of $4\pi G\varrho_c/\zeta_0^2$ as the unit for ω^2 has a decisive role in making the figures look as they do. Had one chosen $4\pi G\varrho_c$ as the unit, for examples, the extrema would have disappeared and the effect would have been reduced to minute changes in slopes and inflections of the curves.

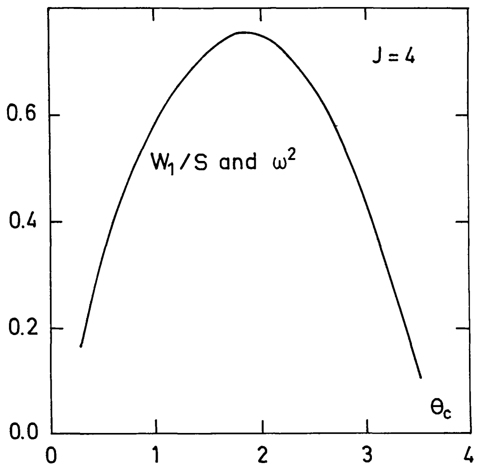

Fig. 4. Same as in Fig. 1 for $j=4$ models. In the adopted scale W_2/S was too small to be shown, and ω^2 and W_1/S coincide

The density waves (i.e., perturbations in the density) were computed according to Eqs. (36) of Paper I. We only report that with increasing j, increasing θ_c, and increasing mode number, they become more and more localized in the vicinity of the center and leave no significant trail in the halo. Evidently this behavior is shared by all physical characteristics associated with the perturbations and is not exclusive to density waves.

5. Discussion

Distributions $F(E)$ for which $dF/dE>0$ for $E<0$ and $F|dF/dE|^{-1/2}=0$ at $E=0$, are stable against all infinitesimal perturbations (Paper I, Theorem). Instabilities are possible if $dF/dE<0$ only, and on account of the W_2 term, the perturbations of the self gravitation. Dorémus, Feix, and Baumann (1971), however, show that the latter distributions are also stable against radial perturbations. Stability of solutions of Eq. (7) is also proved in the appendix. That all ω^2 for $j=1, 2, 3, 4$, and for θ_c values giving finite equilibrium models are real is in perfect accord with the conclusion of Dorémus et al., and with that of the appendix.

The author is not aware of any eigensolutions of the energy-truncated distributions to compare the present results with. Some investigators taking thermodynamical and statistical mechanical approach to cluster problems, however, have discussed the stability of these distributions. Lynden-Bell and Wood (1968) find that in Woolley's models ($j=1$) at $\theta_c \sim 8.5$ ($\theta_c=k$ in their notation) entropy reaches a maximum. They deduce that "...the actual path of evolution departs radically from Woolley's models at this point". Katz (1980) studies Woolley's, Michie-King's, and Peterson's models ($j=1, 2, 3$) in connection with gravothermal instabilities. He reports that the models become unstable at $\theta_c=7.65$, 7.40, and 6.56, respectively. While we do not find instability for these values of θ_c, we do find minima for W_2 and W_1 in the vicinities.

Horwitz and Katz (1977, 1978) consider systems confined to constant volumes and subject to other thermodynamic constraints. They devise grand canonical, canonical, and microcanonical ensembles for their models. They conclude that "...instabilities with respect to spherically symmetric perturbations are associated with a change of sign of thermodynamic quantities". Katz et al. (1978) apply the same technique to Woolley's and King's models and find that (i) in the scheme of microcanonical ensembles Woolley's and King's models become unstable at $\theta_c=7.65$ and 7.40, respectively. (ii) In the scheme of canonical ensembles instabilities develop at $\theta_c=2.45$ and 1.35. In the present computations, the only noteworthy feature is the appearance of minima of W_2 and W_1 at slightly larger θ_c's of case (i) and the appearance of maxima at slightly larger θ_c's of case (ii).

Doubts have been expressed whether the Boltzmann-Liouville equation and the thermodynamic formalism consider the same physical problem. As to the question of stability, the present work does not agree with the conclusions of thermodynamical approach. It does, however, indicate the existence of a common feature between the two problems. This is, the close correspondence of the extrema of W_2 and W_1 with sign changes of thermodynamic quantities of Horwitz et al. Admittedly, the indication is a weak one. But in the absence of any other convincing evidence to settle the dispute it is worth noting it. Exact correspondence of extrema with sign changes of Horwitz and collaborators is not important at this stage. After all, the present work considers only the first order terms in the Fourier expansion of the perturbations. As pointed out in Sect. 3, this is a variational ansatz in the velocity space. It can only overestimate the eigenvalues and make the extrema appear at larger values of θ_c.

Finally, Miller's (1973) remark is noteworthy. He examines Boltzmann's H-theorem and finds that "...the variational calculations to find the single particle ensemble distribution...can violate the n-body equations of motions and the Liouville equation in $6n$ dimensions". The present analysis reports direct solution of the Boltzmann-Liouville equation. Its conclusions seem to be more in line with Miller's conclusions than with the inferences of those who argue that thermodynamic instabilities indicate dynamical instabilities of the Newtonian cluster models.

Appendix: stability of solutions of Eqs. (7) and (9)

Dorémus et al. (1971) show that distributions $F(E)$ for which $dF/dE<0$ are stable against radial perturbations. In considering their multiple water-bag models they implicitly use the transformation (v_r, v_θ, v_ϕ) to (E, J, v_ϕ), where $E=\frac{1}{2}v^2+U$, and $J^2=r^2(v_\theta^2+v_\phi^2)$. Gillon, Cantus, and Baumann prove the stability against radial perturbations for $F(E, J)$ with $dF/dE<0$. In doing so they make an explicit use of the same transformation. The transformation is not a one-to-one mapping. For, the double values $(\pm v_r, \pm v_\theta, v_\phi)$ correspond to the same (E, J, v_ϕ). How the difficulty is circumvented is not clear. In the following we prove the stability against the special class of the radial perturbations we use in this paper. The method is entirely different from those of the authors referred to above, and should serve as an independent verification of their results.

(i) An inequality for W_1: Let Eq. (9b) be written as follows

$$W_1=4\pi[3A^2+6B^2+C^2-2D^2], \tag{A1}$$

where

$$A^2=\int_0^R \Phi^{(4)}\left(\frac{d\xi}{dr}\right)^2 r^2 dr>0, \tag{A2}$$

$$B^2=\int_0^R \Phi^{(4)}\xi^2 dr>0, \tag{A3}$$

$$C^2=4\pi G\int_0^R \varrho\Phi^{(2)}\xi^2 r^2 dr>0, \tag{A4}$$

$$D^2=-\int_0^R \frac{d\Phi^{(4)}}{dr}\xi^2 r dr>0. \tag{A5}$$

Integrating D^2 by parts gives

$$D^2 = B^2 + 2\int \Phi^{(4)} \frac{d\xi}{dr} \xi r dr$$
$$\geqq B^2 + 2\left|\int \Phi^{(4)} \frac{d\xi}{dr} \xi r dr\right|$$
$$\geqq B^2 + 2AB\,, \tag{A6}$$

where the second inequality follows from the Schwartz inequality below

$$\left|\int \Phi^{(4)} \frac{d\xi}{dr} \xi r dr\right|$$
$$\leqq \left[\int \Phi^{(4)} \left(\frac{d\xi}{dr}\right)^2 r^2 dr \int \Phi^{(4)} \xi^2 dr\right]^{1/2} = AB\,. \tag{A7}$$

Substituting Eq. (A7) in Eq. (A1) gives

$$W_1 \geqq 4\pi[2A^2 + (A-2B)^2 + C^2]\,. \quad \text{QED.} \tag{A8}$$

(ii) An inequality for W_2: From Eq. (8b) one has:

$$|\Psi^{(2)}|^2 = \frac{16\pi^2}{9}\left|\int_0^{\sqrt{-2U}} \left(-\frac{dF}{dE}\right)^{1/2} v^2 dv\right|^2$$
$$\leqq \frac{4\pi}{3}\int_0^{\sqrt{-2U}} \left(-\frac{dF}{dE}\right) v^4 dv \, \frac{4\pi}{3}\int_0^{\sqrt{-2U}} v^2 dv\,, \tag{A9}$$

where the inequality is again a Schwartz one. For $F(E)$ one has $v dF/dE = dF/dv$. Substituting this in the first integral on the left side of Eq. (A9) and integrating by parts gives

$$\frac{4\pi}{3}\int\left(-\frac{dF}{dE}\right) v^4 dv = 4\pi\int F v^2 dv = \varrho\,. \tag{A10}$$

By Eq. (8a), the second integral on the left side of Eq. (A9) is $\Phi^{(2)}$. Therefore,

$$|\Psi^{(2)}|^2 \leqq \varrho\Phi^{(2)}\,. \tag{A11}$$

Substituting Eq. (A11) in Eq. (9c) and using the notation of Eq. (A4) gives

$$W_2 < 4\pi C^2\,. \quad \text{QED.} \tag{A12}$$

From Eqs. (A8) and (A12) it now follows

$$W_1 - W_2 > 4\pi|2A^2 + (A-2B)^2| > 0\,. \tag{A13}$$

Thus, all ω^2's of Eq. (9) are positive and the distributions $F(E)$ are stable against the special class of the radial perturbations of this paper.

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