Eigensolutions of Antonov's Equation

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Abstract. Antonov's equation, a linearized version of the collisionless Boltzmann equation for self-gravitating stellar systems, is studied for its symmetries. For an equilibrium system of spherical configuration, the symmetry group is $O(3)$. There exists an angular momentum operator in the six dimensional phase space which commutes with Antonov's operator. This enables one to reduce the solution of Antonov's equation from the six dimensional space to a two dimensional one.

1. Introduction

Collisionless Boltzmann's equation along with that of Poisson has long been used to study the possible equilibrium structures of galaxies and star clusters. It was Antonov (1960), however, who first considered a perturbation version of these equations and addressed the stability of such equilibria. In the phase space ($\mathbf{q}$, $\mathbf{p}$), he separated a perturbation $\phi(\mathbf{q}, \mathbf{p})$ on a distribution function $F(E)$, E is the energy integral, into even and odd components in $\mathbf{p}$ and demonstrated that the dynamics of the odd component is governed by a self adjoint operator in some function space. Throughout sixties to eighties there was a surge of interest in Antonov's approach, but mainly to explore the stability criteria. Among the many attempts one may quote those of Lynden-Bell (1966), Milder (1967), Ipser & Thorne (1968), Lynden-Bell & Sanitt (1969), Sobouti (1984). Stability of anisotropic distributions was addressed by Doremus et al. (1970, 1971), Doremus & Feix (1973), Gillon et al. (1976) and Kandrup and Sygnet (1985). Attempts to obtain the eigensolutions of Antonov's self adjoint operator are those of Sobouti (1984, 1985, 1986) and Sobouti & Samimi (1989, 1995). Here, we further elaborate on the symmetries of Antonov's equation and attempt to extract as much information on its eigensolutions as these symmetries allow.

2. Exposition of the Problem

Let $F(E)$, $E = \frac{1}{2}p^2 + U(q)$, $U(q)$ is a gravitational potential, be an equilibrium distribution function. Let $\phi(\mathbf{q},\mathbf{p},t) = |dF/dE|^{1/2} f(\mathbf{q},\mathbf{p},t)$ be a small perturbation on F. The linearized collisionless Boltzmann and Poisson equation governing the evolution of f is (Sobouti 1984, Sobouti & Samimi 1989)

$$i\frac{\partial f}{\partial t} = \mathcal{A}f = \mathcal{L}f + G\,\mathrm{sign}\,(F_E)|F_E|^{1/2}\mathcal{L}\int \frac{|F'_E|^{1/2}\mathcal{L}'f'}{|\mathbf{q}-\mathbf{q}'|}\,d\mathbf{q}'d\mathbf{p}', \quad (1)$$

where $\mathcal{L} = -i\,(p_j\partial/\partial q_j - \partial U/\partial q_j \partial/\partial p_j)$, $j = 1, 2, 3$, is Liouville's operator in the six-dimensional phase space $(\mathbf{q},\mathbf{p})$. It is multiplied by $i = \sqrt{-1}$ to render it hermitian. The integral represents perturbation in the self-gravitational field and is over the volume of the phase space available to the system. A *"prime"* in the integrand indicates that the quantity in question is to be evaluated at point $(\mathbf{q}', \mathbf{p}')$. The operator $\mathcal{A}$ as defined by the right hand side of Eq. (1) will be called Antonov's operator in honor of his 1960 pioneering on the subject. We decompose f into even and odd terms in $\mathbf{p}$:

$$\begin{aligned} f(\mathbf{q},\mathbf{p}) &= u(\mathbf{q},\mathbf{p}) + iv(\mathbf{q},\mathbf{p}), \\ u(\mathbf{q},\mathbf{p}) &= -u(\mathbf{q},-\mathbf{p}), \\ v(\mathbf{q},\mathbf{p}) &= +v(\mathbf{q},-\mathbf{p}). \end{aligned} \quad (2)$$

Eq. (2) is not a decomposition into real and imaginary parts as yet, though it will turn out to be that as well. Substitution of Eq. (2) in Eq. (1) and noting that both $\mathcal{L}$ and $\mathcal{A}$ are odd in $\mathbf{p}$, gives

$$\frac{\partial u}{\partial t} = \mathcal{A}v, \qquad -\frac{\partial v}{\partial t} = \mathcal{A}u = \mathcal{L}u, \quad (3)$$

from which one obtains

$$-\frac{\partial^2 u}{\partial t^2} = \mathcal{A}^2 u \qquad \text{(Antonov's Equation)}. \quad (4)$$

3. Function Space

Let $\mathcal{H} : \{g(\mathbf{q},\mathbf{p}), (g,g) = \int g^* g\, d\mathbf{q}d\mathbf{p} < \infty\}$ be the Hilbert space of the square integrable complex functions $g(\mathbf{q},\mathbf{p})$. One may readily verify that $\mathcal{L}$ on $\mathcal{H}$ is hermitian: $(g, \mathcal{L}f) = (\mathcal{L}g, f)$, $g, f \in \mathcal{H}$. Antonov's operator, however, is not. For, it is the evolution operator in a time-dependent potential, $U(q) + \delta U(\mathbf{q}, t)$. If, however, one decomposes $\mathcal{H}$ into two subspaces, $\mathcal{H}_\pm : \{g_\pm(\mathbf{q}, \mathbf{p}) = \pm g_\pm(\mathbf{q}, -\mathbf{p})\}$, even and odd in $\mathbf{p}$, one can show that $\mathcal{A}^2$ on $\mathcal{H}_-$ is hermitian. Thus, Eq. (3b) is an eigenvalue equation and may be written as

$$\mathcal{A}^2 u = \omega^2 u. \tag{5}$$

Naturally, ω^2 is real and any two eigenfunctions u and u' belonging to distinct ω and ω' are orthogonal. One further remark: $\mathcal{A}$ is purely imaginary and $\mathcal{A}^2$ is real. Therefore, u of Eq. (4) can be chosen real. Eq. (3) then shows that v also is real. Thus, the decomposition of Eq. (2) is a decomposition into real and imaginary components as well.

4. Symmetries of $\mathcal{L}$ and $\mathcal{A}$

4.1. Discrete symmetries

Let Q and P be two parity operators on functions of $\mathbf{q}$ and $\mathbf{p}$, respectively:

$$Qg(\mathbf{q}, \mathbf{p}) = g(-\mathbf{q}, \mathbf{p}), \qquad Pg(\mathbf{q}, \mathbf{p}) = g(\mathbf{q}, -\mathbf{p}). \tag{6}$$

For an even potential, $U(\mathbf{q}) = U(-\mathbf{q})$, both $\mathcal{L}$ and $\mathcal{A}$ are odd in both $\mathbf{q}$ and $\mathbf{p}$. One as the following anticommutation and commutation rules

$$\begin{array}{ll} \{\mathcal{L}, Q\} = \{\mathcal{L}, P\} = 0, & [\mathcal{L}, QP] = 0, \\ \{\mathcal{A}, Q\} = \{\mathcal{A}, P\} = 0, & [\mathcal{A}, QP] = 0, \end{array} \tag{7}$$

$$[\mathcal{A}^2, Q] = [\mathcal{A}^2, P] = 0. \tag{8}$$

One, therefore, may choose solutions $u(\mathbf{q}, \mathbf{p})$ either odd or even in either $\mathbf{q}$ or $\mathbf{p}$. By Eq. (3), then the parities of $v(\mathbf{q}, \mathbf{p})$ will be opposite to those of $u(\mathbf{q}, \mathbf{p})$.

4.2. Continuous symmetries, $O(3)$ symmetry

For $F(E)$, $U(q)$ is spherically symmetrical. Simultaneous rotations of q and p axes about the same direction by the same angle, leaves $\mathcal{L}$ and $\mathcal{A}$ form invariant. The generators of such infinitesimal rotations in $\mathcal{H}$ are the following angular momentum operators

$$J_i = L_i + K_i = i\varepsilon_{ijk}(q_j \partial/\partial q_k + p_j \partial/\partial p_k). \tag{9}$$

This is the sum of two angular momenta, L_i in q space and K_i in p space. Their algebra is

$$[J_i, J_j] = -i\varepsilon_{ijk} J_k, \quad [L_i, L_j] = -i\varepsilon_{ijk} L_k, \quad [K_i, K_j] = -i\varepsilon_{ijk} K_k. \quad (10)$$

By direct verification one may easily see that

$$[\mathcal{L}, J_i] = [\mathcal{A}, J_i] = [\mathcal{A}^2, J_i] = 0. \quad (11)$$

There follows that it is possible to find simultaneous eigenfunctions for the set of operators $(\mathcal{A}^2, J^2, J_z, Q, P)$ and $(\mathcal{A}, J^2, J_z, QP)$. Simplifications that these symmetries introduce are enormous. It will be explained shortly that the eigensolutions of J^2 and J_z can be worked out in terms of the direction angles of $\mathbf{q}$ and $\mathbf{p}$ vectors. This immediately reduces the six-dimensional integro-differential Eq. (4) into a two-dimensional one in terms of the magnitudes of q and p.

5. Eigensolutions J^2 and J_z

From the literature in angular momentum one learns that it is possible to find simultaneous eigenfunctions for (J^2, J_z, L^2, K^2). Let, in *bra* and *ket* notation of Dirac, this be $|jmlk\rangle$ with respective eigenvalues $j(j+1)$, $-j \leq m < j$, $l(l+1)$ and $k(k+1)$. One also learns that there exists simultaneous eigenfunctions for (L^2, L_z, K^2, K_z). This simply is $|lm_l km_k\rangle = Y_l^{m_l}(\theta, \phi) Y_k^{m_k}(\alpha, \beta)$, where (θ, ϕ) are the direction angles of $\mathbf{q}$ and (α, β) are those of $\mathbf{p}$. Both sets of kets are complete and can be expressed in terms of one another. Thus

$$|jmlk\rangle = \sum_{m_l} |lm_l km_k\rangle\langle lm_l km_k|jmlk\rangle, \qquad m_k = m - m_l, \quad (12)$$

where $\langle \ldots | \ldots \rangle$ are the Clebsch-Gordan coefficients, subject to the restrictions $m = m_l + m_k$ and (j, l, k) satisfying the triangle condition, the sum of any two to be larger than the other and the difference of any two to be smaller than the other. With these preliminaries one may now cast Eq. (4) into a variational form. First we multiply it by $u^*(\mathbf{q}, \mathbf{p})$ and integrate over the volume of phase space. Noting the hermiticity of $\mathcal{L}$ we obtain

$$\begin{aligned} \omega^2(u, u) &= (u, \mathcal{A}^2 u) = (\mathcal{A}^\dagger u, \mathcal{A} u) = \\ &\quad (\mathcal{L}u, \mathcal{L}\mathcal{P}u) + G\,\mathrm{sign}\,(F_E) \times \\ &\quad \int |F_E|^{1/2} (\mathcal{L}u)^* |F'_E|^{1/2} (\mathcal{L}u)' |\mathbf{q} - \mathbf{q}'|^{-1}\, d\mathbf{q} d\mathbf{p} d\mathbf{q}' d\mathbf{p}'. \end{aligned} \quad (13)$$

Next, for a given j and m, we let

$$u(\mathbf{q}, \mathbf{p}) = \sum_{l,k} u_{lk}(q,p)|jmlk>, \tag{14}$$

where (j, l, k) is subject to triangle condition. We substitute Eq. (11) into Eq. (10) and integrate over the direction angles (θ, ϕ) and (α, β) appearing in $|jmlk>$. Finally, we adopt a polynomial expression in q and p for $u_{lk}(q,p)$ and go through the machinery of variational calculations. This converts the variational integral Eq. (13) into a set of algebraic equations for the coefficients appear in the polynomial expression. Vanishing of the characteristic determinant will give the eigenvalues. For radial oscillations of polytropes, $j = 0$, some periods of oscillations are reported by Samimi & Sobouti (1995).

6. Concluding Remarks

- Evolution of the odd-p component of a small perturbation on a distribution $F(E)$ is governed by a self adjoint operator $\mathcal{A}^2$, Eq. (4).
- $\mathcal{A}^2$ has an $O(3)$ symmetry and can have simultaneous eigenfunctions with J^2 and J_z.
- Eigenfunctions of J^2 and J_z can be expressed in terms of the products of two spherical harmonics $Y_l^{m_l}(\theta, \phi)$ and $Y_k^{m_k}(\alpha, \beta)$. This reduces the problem from the six-dimensional space to a two-dimensional one in terms of the magnitudes of q and p.
- Eigensolutions of $\mathcal{A}^2$ are classified into sequences designated by two discrete indices j and m.
- Within each (j, m) sequence discrete eigenfrequencies have been found for polytropes via a variational scheme.
- The question of whether the eigenfrequencies of Antonov's equation are discrete or continuous or both remains open. It may be worth knowing that, however, for a quadratic potential and neglecting the perturbations in self-gravitation, Antonov's problem can be solved exactly (Sobouti & Dehghani 1992). The modes are discrete and each is designated by six integers.
- A last point on the same issue: hydrodynamic equations, continuity and Euler's have discrete eigenfrequencies. They are the

zeroth and the first p-moments of Eq. (1) in which the second moment is, through an equation of state, expressed in terms of the zeroth one and the sequence is terminated. Isn't it feasible that if one considers n of such moment equations and expresses the $n + 2^{\text{nd}}$ moment in terms of the n-th one, one will obtain discrete eigensolutions. In star clusters and galaxies collisions do not play as decisive a role an in a gas, for instance. Isn't it, however, possible that they might be operative in smearing out the higher p moments of the distribution function. If this takes place the system will have discrete eigenfrequencies. Variational calculations which deal with a finite number of trial functions just do such smearings. All these considerations are indicative of discrete oscillation modes in stellar systems. The question of whether the present precision of the photometric CCD records allow their detection is, of course, a matter of the art observation.

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