Astron. Astrophys. 210, 18–24 (1989)

ASTRONOMY AND ASTROPHYSICS

Liouville's equation

I. Symmetries and classification of modes*

Y. Sobouti

Department of Physics and Biruni Observatory, College of Sciences, Shiraz University, Shiraz, Iran

Received September 28, 1987; accepted May 16, 1988.

Summary. Let Q and P be parity operators in configuration and momentum spaces, respectively. Let $\boldsymbol{L}$ and $\boldsymbol{K}$ be angular momentum operators also in configuration and momentum spaces. Let $\mathscr{L}$ be the Liouville operator and $\boldsymbol{J} = \boldsymbol{L} + \boldsymbol{K}$. We show that (1) $\{\mathscr{L}, J^2, J_z, QP\}$ commute mutually and (2) the operators $\{\mathscr{L}^2, J^2, J_z, Q, P\}$ commute mutually. This enables one to classify (a) the eigenmodes of Liouville's equation to classes which are simultaneously eigenfunctions of $\{J^2, J_z, QP\}$ and (b) those of $\mathscr{L}^2$ to classes which are simultaneously eigenfunctions of $\{J^2, J_z, Q, P\}$. The commutation relations also simplify mode calculations. In a six dimensional phase space, by expanding eigenfunctions of $\mathscr{L}$ and $\mathscr{L}^2$ in terms of those of the remaining commuting operators, the problem reduces to a two dimensional one. By way of illustration this reduction is worked out in detail for the class of modes belongging to *(0, 0, even, even)* eigenvalues of $\{J^2, J_z, Q, P\}$.

Key words: Galaxies – dynamics and evolution – Liouville's equation – symmetries and normal modes

1. Introduction

In their study of the statistics of stellar systems, astrophysicists have popularly used the Liouville equation (or at least a six dimensional version of it) as a working tool. Time-independent solutions in the form of functions of the integrals of motion have earned wide acceptance. Stellar hydrodynamics as moments of the Liouville equation have served some purpose. The past thirty years have also seen perturbation versions of it in Antonov's presentation (1960), and in the form of density wave theory in flattened disks (Lin et al., 1969; Shu, 1970). Lynden-Bell (1962, 1967) has discussed initial value problems for small disturbances in encounterless stellar systems and has proposed damping of small scale disturbances as the stars become well mixed. Barnes et al.'s (1986) analysis is noteworthy in that they place emphasis on some group properties of the linearized equation. One's knowledge of the time-dependent solutions of Liouville's equation, however, is meager. Even in linear regimes (cases of time constant potentials or linear perturbation) one knows of no explicit time-dependent solutions.

In this paper we point out some of the symmetries of Liouville's equation and aim at the construction of exact eigensolutions. For pedagogical reasons here we consider time-independent potentials. This is a step toward a more extensive analysis of realistic self-gravitating systems where potentials vary in time. It turns out that perturbations of realistic systems are governed by an integro-differential operator with most of the symmetries of the time constant Liouville operator. This, however, will be presented elsewhere.

In Sect. 2 we review the existing literature on time-dependent solutions. The material is collected mainly from sources on statistical mechanics. We conclude that the eigenfunctions of Liouville's operator are complex functions of phase coordinates in a complex Hilbert space. The latter in turn is the direct product of two Hilbert spaces, one accommodating functions of configuration coordinates and the other those of the momentum coordinates. In Sect. 3 we introduce a pair of parity operators, Q and P, in configuration and momentum spaces, respectively. We show that QP commutes with the Liouville operator, $\mathscr{L}$, and classifies the latter's eigenfunctions on the basis of their parities. In Sect. 4 we introduce a pair of angular momentum operators $\boldsymbol{L}$ and $\boldsymbol{K}$ in configuration and momentum spaces, respectively. We show that $\boldsymbol{J} = \boldsymbol{L} + \boldsymbol{K}$ commutes with $\mathscr{L}$ and QP. In Sect. 5 we construct the simultaneous eigenfunctions of the mutually commuting operators $\mathscr{L}$, J^2, J_z, and QP. This (a) provides a classification scheme for the normal modes of Liouville's operator, and (b) reduces the six dimensional phase space problem to a two dimensional one in terms of the magnitudes of the position and momentum vectors. In Sect. 6 we present simple examples of low order modes. Section 7 is devoted to concluding remarks.

Evidently the pairs of parity and angular momenta operators presented here are examples from a host of other pairs which act on phase space functions. Prigogine and his coworkers call them superoperators. They give their own examples and attempt to develop a 'superoperator' viewpoint of statistical mechanics (Prigogine, 1980; George and Prigogine, 1979).

2. Review of basic principles

A formal exposition of microscopic and macroscopic dynamical quantities, their interrelations and time evolutions may be found in Balescu (1975, pp. 37–44). Here, for pedagogical reasons, we confine ourself to a six dimensional phase space and to systems subjected to time-independent potentials, though most of the conclusions are valid in 6N dimensional spaces. The Liouville

* Contribution No. 15, Biruni Observatory

equation may be written as

$$i\frac{\partial F}{\partial t} = \mathscr{L}F, \qquad \mathscr{L} = -i\left(p_i\frac{\partial}{\partial q_i} - \frac{\partial U}{\partial q_i}\frac{\partial}{\partial p_i}\right), \tag{1}$$

where $(\boldsymbol{q}, \boldsymbol{p})$ are configuration and momentum coordinates, $U(\mathbf{q})$ is a potential, $F(\boldsymbol{q}, \boldsymbol{p}, t)$ is a phase space function, and $\mathscr{L}$ is the Liouville operator. In order to be a probability density, F is required to be positive everywhere in the phase space and be normalized to unity:

$$F(\boldsymbol{q}, \boldsymbol{p}, t) \geq 0 \quad \text{for all } \boldsymbol{q},\ \boldsymbol{p} \text{ and } t, \tag{2}$$

$$\int F(\boldsymbol{q}, \boldsymbol{p}, t)\, d\boldsymbol{q}\, d\boldsymbol{p} = 1 \quad \text{for all } t. \tag{3}$$

2.1. *Hilbert space of the phase space functions*

An axiomatic study of the eigensolutions of Eq. (1) requires introduction of a function space. Let H be the space of all complex functions $f(\boldsymbol{q}, \boldsymbol{p}) = u(\boldsymbol{q}, \boldsymbol{p}) + iv(\boldsymbol{q}, \boldsymbol{p})$, (a) satisfying the Cauchy convergence condition

$$(f, f) = \int f^* f\, d\boldsymbol{q}\, d\boldsymbol{p} = \text{finite}, \quad \text{for all } f \text{ in } H, \tag{4}$$

and (b) vanishing at the boundary of the phase space,

$$f(\boldsymbol{q}, \boldsymbol{p}), u(\boldsymbol{q}, \boldsymbol{p}), v(\boldsymbol{q}, \boldsymbol{p}) = 0 \text{ at boundary}. \tag{5}$$

If the phase space is infinite, condition (4) will ensure the vanishing of f, u and v at infinity. Condition (5) then becomes redundant. In conformity with Eq. (4) the space will be endowed with the inner product

$$(f, g) = \int f^* g\, d\boldsymbol{q}\, d\boldsymbol{p} = \text{finite} \quad \text{for all } f \text{ and } g \text{ in } H. \tag{6}$$

There is a large body of literature on the Hilbert space formulation of phase space problems (see, among others, Schonberg, 1952 and Prigogine, 1962). The eigenfunctions of Liouville's operator find a natural place in such a space and constitute a complete orthonormal set.

2.2. *The eigenvalue problem*

The material in Sects. 2.2 and 2.3 are mainly from Prigogine (1962). The results are written as numbered theorems for easy reference in subsequent sections.

Theorem 2.1: $\mathscr{L}$ is Hermitian. This is proved by showing that $(g, \mathscr{L}f) = (\mathscr{L}g, f)$, for all f and g in H, by integrations by parts and letting the integrated terms vanish by Eq. (5).

From Theorem 2.1 follows the eigenvalue problem

$$\mathscr{L}f = \omega f, \tag{7}$$

which in turn implies the existence of time-dependent solutions of Eq. (1) of the form $f(\boldsymbol{q}, \boldsymbol{p}) \exp(-i\omega t)$.

Theorem 2.2: ω's are real and the eigenfunctions belonging to distinct ω's are orthogonal. Proof follows from the Hermitian character of $\mathscr{L}$. Orthogonality of two functions f and g is in the sense of Eq. (6), $(f, g) = 0$.

Completeness of the set of eigenfunctions: For a harmonic potential, $U = \frac{1}{2}kr^2$, or a potential well, $U =$ *constant* in some domain and infinite outside the domain, completeness of the eigenfunctions is proved in Appendix. For more general potentials we conjecture it to be true.

2.3. *Non-zero eigenvalues*

The Liouville operator is purely imaginary, $\mathscr{L}^* = -\mathscr{L}$. There follows:

Theorem 2.3: (a) Eigenfunctions belonging to $\omega \neq 0$ are complex. (b) If a pair (ω, f) is an eigensolution, then
(1) $(-\omega, f^*)$ is another eigensolution
(2) $f^* f$ is an integral of motion
(3) $[(n-m)\omega, f^{*m} f^n]$ is an eigensolution, $n, m =$ positive integers.

Proof: (a) $\mathscr{L}$ is imaginary and ω is real. Equation (7) can hold if and only if f is complex. The proof breaks down if $\omega = 0$.
(b1) $-\mathscr{L}^* f^* = \mathscr{L} f^* = -\omega f^*$.
(b2) $\mathscr{L}(f^* f) = (\mathscr{L} f^*) f + f^*(\mathscr{L} f) = (\omega - \omega) f = 0$, for $\mathscr{L}$ is a linear first order differential operator.
(b3) also follows from the linearity of $\mathscr{L}$. Restriction of n and m to positive integers is to ensure the analyticity of $f^{*m} f^n$, i.e. the existence of all Cauchy derivatives. QED.

Parts *a*, *b1* and *b2* of this theorem are due to Prigogine. Part *b3* is a generalization of *b2*. Alternative forms of Eq. (7) may be obtained by decomposing f into real and imaginary parts. Thus,

$$f = u(\boldsymbol{q}, \boldsymbol{p}) + iv(\boldsymbol{q}, \boldsymbol{p}). \tag{8}$$

Substituting Eq. (8) in Eq. (7) and separating the real and imaginary parts gives

$$\mathscr{L}u = i\omega v, \tag{9}$$

$$\mathscr{L}v = -i\omega u. \tag{10}$$

One also obtains

$$\mathscr{L}^2 u = \omega^2 u, \tag{11}$$

$$\mathscr{L}^2 v = \omega^2 v. \tag{12}$$

Theorem 2.4: Eigenfunctions belonging to $\omega \neq 0$ integrate to zero (Prigogine, 1962).

$$\int f\, d\boldsymbol{q}\, d\boldsymbol{p} = 0, \qquad \omega \neq 0. \tag{13}$$

Proof: From Eq. (7), $\int f\, d\boldsymbol{q}\, d\boldsymbol{p} = \omega^{-1} \int \mathscr{L} f\, d\boldsymbol{q}\, d\boldsymbol{p} = 0$, by integrations by parts. QED. This integrability to zero holds for u and v separately.

2.4. *General form of the time-independent and time-dependent distribution functions*

An important corrollary to Theorem 2.4 is the following: No eigenfunction belonging to a non-zero eigenvalue nor any linear combinations of them can serve as a distribution function and be interpreted as a probability density; for they will not satisfy conditions (2) and (3). To avoid any future confusion we shall reserve the words 'distribution functions' or simply 'distributions' or 'probability densities' for those functions of phase space which are positive everywhere and are integrable to unity. The rest, including all eigensolutions belonging to $\omega \neq 0$, will simply be referred to as phase space functions.

2.4.1. Time-independent distributions

Any solution $F_0(\boldsymbol{p}, \boldsymbol{q})$ satisfying

$$\mathscr{L}F_0 = 0 \tag{14}$$

is an integral of motion. Among the celebrated ones is the energy integral, $E = \frac{1}{2}p^2 + U(\boldsymbol{q})$, or any arbitrary but analytic function $f(E)$. Analyticity here implies infinite differentiability, though for practical convenience one may be contented with the existence of the first few derivatives. If U is axially symmetric then the component of angular momentum along the symmetry axis is another integral of motion. If the potential is spherically symmetric then any component of angular momentum and any arbitrary analytic function of them are integrals of motion. The choice should of course be limited to integrability to one and consequently to square integrability which is the requirement for membership in H.

According to Theorem 2.3*b*, however, any f^*f, where f is an eigensolution, satisfies Eq. (14). This, combined with the completeness of the eigenfunctions, implies that any F_0 should be expansible as follows

$$F_0(\boldsymbol{q},\boldsymbol{p}) = \sum_{\omega} a_\omega f^*_\omega f_\omega\,, \tag{15}$$

where (ω, f_ω) is an eigensolution and a_ω's are constants. Assuming that f_ω's are normalized, Eq. (2) and (3) gives

$$a_\omega \geq 0\,, \quad \text{and} \sum_{\omega} a_\omega = 1\,. \tag{16}$$

Equation (15) should also hold for the energy and angular momentum integrals as well. This is demonstrated explicitly for a harmonic potential in the Appendix.

2.4.2. Time-dependent distributions

A distribution $F(\boldsymbol{q},\boldsymbol{p},t)$ is to be integrable to unity, be real and positive everywhere in the phase space, and be expandable in terms of the complete set of the eigenfunctions $\{f_\omega\}$. Thus, it must have the following form

$$F(\boldsymbol{q},\boldsymbol{p},t) = F_0(\boldsymbol{q},\boldsymbol{p}) + \sum_{\omega} [b_\omega f_\omega e^{i\omega t} + b^*_\omega f^*_\omega e^{-i\omega t}]\,, \tag{17}$$

where F_0 is given by Eq. (15) and is an integral of motion, and b_ω's are expansion constants.

In perturbation problems it is customary to consider distributions of the form $F = F_0 + f(\boldsymbol{q},\boldsymbol{p},t)$ and assume $f \ll F_0$. Expansion (17) resembles this kind of separation into a steady state and a time-varying term. There is, however, a conceptual difference of prime importance. Time-dependent terms of Eq. (17) are not required to remain small in comparison with the time-independent one. There is no approximation involved in Eq. (17). Condition of positive probability at most prevents the time-varying terms from exceeding F_0.

This section has enabled us to write recipes for the time-dependent distributions in terms of eigenfunctions, has disclosed the complex nature of the eigenfunctions and their integrability to zero. More symmetries of $\mathscr{L}$ and further information on eigenfunctions is discussed in Sect. 3.

3. Parity operators

Let Q and P be parity operators defined in q and p spaces, respectively. Thus,

$$Qf(\boldsymbol{q},\boldsymbol{p}) = f(-\boldsymbol{q},\boldsymbol{p})\,, \tag{18}$$

$$Pf(\boldsymbol{q},\boldsymbol{p}) = f(\boldsymbol{q},-\boldsymbol{p})\,. \tag{19}$$

It is elementary to show that Q and P are Hermitian and have two eigenvalues ± 1. Eigenfunctions of Q or P belonging to $+1$ are even functions of $\boldsymbol{q}$ or $\boldsymbol{p}$, respectively, and those belonging to -1 are odd functions. The sets of even and odd functions are complete.

At this stage we assume the potential to be symmetric in $\boldsymbol{q}$, $U(\boldsymbol{q}) = U(-\boldsymbol{q})$. There follows

Theorem 3.1:

(1) Q and P commute, $[Q,P] = 0$.
(2) Q and P anticommute with $\mathscr{L}$, $\{Q,\mathscr{L}\} = 0$, and $\{P,\mathscr{L}\} = 0$.
(3) QP commutes with $\mathscr{L}$, $[QP,\mathscr{L}] = 0$.
(4) Q and P commutes with $\mathscr{L}^2$, $[Q,\mathscr{L}^2] = 0$ and $[P,\mathscr{L}^2] = 0$.

The proof is elementary and follows from the fact that $\mathscr{L}$ is odd both in $\boldsymbol{q}$ and $\boldsymbol{p}$.

Notation: If necessary the parities of a function $u(\boldsymbol{q},\boldsymbol{p})$ will be indicated by a pair of subscripts '*e*' and '*o*' (for even and odd). The first subscript will indicate the q parity and the second the p parity. For example $u_{eo}(\boldsymbol{q},\boldsymbol{p})$ will be even in q and odd in p; $u_{ee}(\boldsymbol{q},\boldsymbol{p})$ will be even in both, etc.

Theorem 3.2: QP is a parity operator in H. Its eigenfunctions belonging to $+1$ are of the form u_{ee} and u_{oo}. Those belonging to -1 are of the form u_{eo} and u_{oe}. The proof is elementary.

Let u and v be the real and imaginary parts of an eigenfunction f, Eq. (8). One has

Theorem 3.3: The q and p parities of u are opposite to those of v.

Proof: From Eq. (11) u is an eigenfunction of $\mathscr{L}^2$. Since $[\mathscr{L}^2, Q] = 0$, u can be chosen even (odd) in $\boldsymbol{q}$. From Eq. (9), $v = (i\omega)^{-1}\mathscr{L}u$. Since $\mathscr{L}$ is odd in q and u is chosen even (odd), v has to be odd (even) in $\boldsymbol{q}$. This proves the opposite parities of u and v as regards $\boldsymbol{q}$. The argument can be repeated for $\boldsymbol{p}$. QED.

Corrollary to Theorems 3.2 and 3.3: With no loss of generality the eigenfunctions of $\mathscr{L}$ can be chosen as either

$$f = u_{ee} \pm iv_{oo}, \text{ even parity of } QP\text{, or} \tag{20}$$

$$f = u_{oe} \pm iu_{eo}, \text{ odd parities of } QP\,. \tag{21}$$

It is not necessary to consider $u_{oo} \pm iv_{ee}$ or $u_{eo} \pm iv_{oe}$, as these expressions can be brought to previous forms by multiplying by $\pm i$. For both f's given in Eqs. (20) and (21) the corresponding macroscopic space density is associated with u, $\rho = \int (u_{ee}$ or $u_{oe})\,d\boldsymbol{p}$, and the macroscopic velocity field is associated with v, $\rho v = \int (v_{oo}$ or $v_{eo})\boldsymbol{p}\,d\boldsymbol{p}$.

This section has taken us one more step forward in the construction of eigenfunctions. More symmetries of $\mathscr{L}$ is discussed in Sect. 4.

4. Angular momentum operators

Let $\boldsymbol{L}$ and $\boldsymbol{K}$ be two angular momentum operators defined in q and p subspaces of H:

$$L_i = -i\left(q_j \frac{\partial}{\partial q_k} - q_k \frac{\partial}{\partial q_j}\right), \tag{22}$$

$$K_i = -i\left(p_j \frac{\partial}{\partial p_k} - p_k \frac{\partial}{\partial p_j}\right), \tag{23}$$

(i,j,k) = even permutations of $(1,2,3)$.

They satisfy the angular momentum algebra:

$$[L_i, L_j] = -iL_k, (i, j, k) = \text{even perm}\,(1,2,3), \tag{24}$$

$$[K_i, K_j] = -iK_k, (i, j, k) = \text{even perm}\,(1,2,3), \tag{25}$$

$$[L_i, K_j] = 0. \tag{26}$$

In spherically symmetric potentials one can show that

$$[L_i, \mathscr{L}] = -[K_i, \mathscr{L}] = -\left(p_j \frac{\partial}{\partial q_k} - p_k \frac{\partial}{\partial q_j}\right) - \left(\frac{\partial U}{\partial q_j}\frac{\partial}{\partial p_k} - \frac{\partial U}{\partial q_k}\frac{\partial}{\partial p_j}\right),$$

$$(i, j, k) = \text{even perm}\,(1,2,3). \tag{27}$$

The proof is a matter of straightforward calculation. We only hint that in the case of spherical symmetry one has

$$\frac{\partial U}{\partial q_l} = \frac{1}{q}\frac{dU}{dq} q_l, \tag{28}$$

$$\frac{\partial^2 U}{\partial q_l \partial q_k} = \frac{1}{q}\frac{dU}{dq}\delta_{lk} + \frac{1}{q}\frac{d}{dq}\left(\frac{1}{q}\frac{dU}{dq}\right) q_l q_k, \tag{29}$$

Definition: $J_i = L_i + K_i, \quad i = 1,2,3.$ (30)

Theorem 4.1:

(1) $\boldsymbol{J}$ is an angular momentum operator
(2) $\boldsymbol{J}$ commutes with $\mathscr{L}$, $[J_i, \mathscr{L}] = 0$.

Proof: Part (1) follows from Eqs. (24)–(26). Part (2) follows from Eq. (27). QED.

Theorem 4.2:

(1) The operators $\{\mathscr{L}, J^2, J_z, QP\}$ commute mutually.
(2) The operators $\{\mathscr{L}^2, J^2, J_z, Q, P\}$ commute mutually.

Proof:

(1) $[\mathscr{L}, J^2] = J_i[\mathscr{L}, J_i] + [\mathscr{L}, J_i]J_i = 0$, by Theorem 4.1.2.

$[\mathscr{L}, J_z] = 0$, by Theorem 4.1.2.

$[\mathscr{L}, QP] = 0$, by Theorem 3.1.3.

$[J^2, J_z] = 0$, by Theorem 4.1.1.

$[J^2, QP] = 0$ and $[J_z, QP] = 0$ by the fact that all components of J are even both in $\boldsymbol{q}$ and $\boldsymbol{p}$.

(2) Mutual commutations of the second set can similarly be inferred from Theorems 3.1, 4.1 and 4.2.1. QED.

Consequences of Theorem 4.2 reach far. The theorem enables one to construct simultaneous eigenfunctions of $\{\mathscr{L}, J^2, J_z, QP\}$ and to construct the real and imaginary parts, u and v, as simultaneous eigenfunctions of $\{\mathscr{L}^2, J^2, J_z, Q, P\}$.

Eigenvalues and eigenfunctions of J^2 and J_z:

The operator $\boldsymbol{J}$ is the vector sum of two independent angular momenta $\boldsymbol{L}$ and $\boldsymbol{K}$ in two diffferent spaces $\boldsymbol{q}$ and $\boldsymbol{p}$. Standard texts in quantum mechanics, spectroscopy, and nuclear interactions discuss the coupling of angular momenta in details. In compiling the following review we have consulted Brink and Satchler (1968), Rose (1957), Condon and Shortley (1935), and Rotenberg et al. (1959).

The four operators $\{J^2, J_z, L^2, K^2\}$ commute mutually. In Dirac's notation let $|jmlk\rangle$ be their simultaneous eigenfunctions. One has

$$J^2|jmlk\rangle = j(j+1)|jmlk\rangle, \quad j = 0,1,2,\ldots \tag{31}$$

$$J_z|jmlk\rangle = m|jmlk\rangle, \quad -j \le m \le j \tag{32}$$

$$L^2|jmlk\rangle = l(l+1)|jmlk\rangle, \quad l = 0,1,2,\ldots \tag{33}$$

$$K^2|jmlk\rangle = k(k+1)|jmlk\rangle, \quad k = 0,1,2,\ldots \tag{34}$$

The operators $\{L^2, L_z, K^2, K_z\}$ also commute mutually. Their simultaneous eigenfunctions are $|lm_l km_k\rangle = Y_{lm_l}(\theta, \varphi) Y_{km_k}(\alpha, \beta)$, where (θ, φ) are the polar angles of $\boldsymbol{q}$, (α, β) are those of $\boldsymbol{p}$, and Y's are spherical harmonics. Both sets of eigenfunctions are complete and one can be expanded in terms of the other. For example

$$|jmlk\rangle = \sum_{m_l, m_k} |lm_l km_k\rangle\langle lm_l km_k|jmlk\rangle, \tag{35}$$

where $\langle\ldots|\ldots\rangle$ are the Clebsch-Gordan coefficients. They are nonzero if (*a*) $m = m_l + m_k$, and (*b*) (j, l, k) satisfy the triangle conditions, that is, the sum of any two is larger than the other and the difference of any two is smaller than the other. Condition (*a*) reduces the double sum in Eq. (35) to a single one, the sum over m_l or m_k, whichever may have the smaller range. Condition (*b*) makes an eigenvalue $j(j+1)$ infinitely degenerate. For the permissible values of j turn out to be

$$j = |l-k|, |l-k|+1, \ldots, l+k. \tag{36}$$

A given j can be constructed by choosing l and k in infinitely many ways. For example, $j = 0$ is constructed by $l = k = 0,1,2,\ldots$, and $j = 1$ by $l = k - 1 = 0,1,2,\ldots$, and by $l = k + 1 = 1,2,\ldots$

The q or p parity of $|jmlk\rangle$ is that of l or k, respectively. Classified on this basis, $|jmlk\rangle$'s also become eigenfunctions of Q and P as well.

5. Normal modes of $\mathscr{L}$: classification and calculation

An eigenfunction f consists of its real and imaginary parts, u and v, each with definite q and p parities indicated in Eqs. (20) and (21). The real part u is a solution of Eq. (11), and v can be calculated from Eq. (9) once u is known. One also has the option to solve Eq. (12) for v and then Eq. (10) for u.

By Theorem 4.2.2, u can be a simultaneous eigenfunction of $\{J^2, J_z, Q, P\}$ as well as $\mathscr{L}^2$. This means that $u(\boldsymbol{q}, \boldsymbol{p})$ can be expanded in terms of the appropriate parity classes of the degenerate $|jmlk\rangle$'s with expansion coefficients depending only on the magnitudes q and p. Thus,

$$u(\mathbf{q}, \mathbf{p}) = \sum_{l,k} u_{lk}(\boldsymbol{q}, \boldsymbol{p}) = \sum_{l,k} |jmlk\rangle \bar{u}_{lk}(q, p). \tag{37}$$

Equation (37) is written for specified values of j, m, q parity, and p parity. All quantities $u(\mathbf{q}, \mathbf{p})$, $u_{lk}(\mathbf{q}, \mathbf{p})$, $\bar{u}_{lk}(q, p)$, and $|jmlk\rangle$ carry these specification. They, however, are suppressed for brevity. The sums over l and k are over the permissible values of Eq. (36).

5.1. Classification of modes

An important part of our goal is achieved. Theorem 4.2 and Eq. (37) provides a classification scheme for the normal modes of Liouville's equation:

"A specified set of (j, m, q parity, p parity) specifies a class of modes".

5.2. Calculation of modes

A mode $u(\boldsymbol{q},\boldsymbol{p})$ of Eq. (37) is determined by giving $\bar{u}_{lk}(q,p)$'s. Substituting expansion (37) in Eq. (11) provides the necessary relations for obtaining these functions. Thus, the six dimensional phase space problem of finding $u(\boldsymbol{q},\boldsymbol{p})$ is reduced to a two dimensional problem of solving for $\bar{u}_{lk}(q,p)$. We shall use a variational technique for this purpose. Before doing so, however, let us write $\mathscr{L}$ in terms of the spherical polar coordinates of $\boldsymbol{q}$ and $\boldsymbol{p}$, for we have already used these coordinates in the construction of $|jmlk\rangle$, Eq. (35). One obtains

$$\mathscr{L} = c\bar{\mathscr{L}} + \frac{p}{q}\mathscr{L}_L + \frac{dU}{dq}\frac{1}{p}\mathscr{L}_k, \tag{38}$$

where

$$c = \cos(\boldsymbol{q},\boldsymbol{p}) = \cos\theta\cos\alpha + \sin\theta\sin\alpha\cos(\varphi-\beta), \tag{39}$$

$$c_\pm = L_\pm c = -K_\pm c, \quad L_\pm = L_x \pm iL_y, \quad \text{etc.}, \tag{40}$$

$$c_z = L_z c = -K_z c, \tag{41}$$

$$\bar{\mathscr{L}} = -i\left(p\frac{\partial}{\partial q} - \frac{dU}{dq}\frac{\partial}{\partial p}\right), \tag{42}$$

$$\mathscr{L}_L = \tfrac{1}{2}i[c_+L_- + c_-L_+ + 2C_zL_z], \tag{43}$$

$$\mathscr{L}_K = \tfrac{1}{2}i[c_+K_- + c_-K_+ + 2c_zK_z]. \tag{44}$$

We note that $\bar{\mathscr{L}}$ is in terms of the magnitudes of $\boldsymbol{q}$ and $\boldsymbol{p}$, and $\mathscr{L}_L$ and $\mathscr{L}_K$ are in terms of the angular coordinates of $\boldsymbol{q}$ and $\boldsymbol{p}$. $\bar{\mathscr{L}}$ is Hermitian in a (q,p) subspace of the original Hilbert space, H. The inner product in this subspace is

$$(\bar{u},\bar{u}') = \int \bar{u}^*\bar{u}' q^2\,dq\,p^2\,dp. \tag{45}$$

The angle operators $\mathscr{L}_L$ and $\mathscr{L}_K$ are Hermitian in a $(\theta,\varphi;\alpha,\beta)$ subspace of H. The inner product in this subspace is

$$\langle\psi,\psi'\rangle = \int \psi^*\psi' \sin\theta\,d\theta\,d\varphi\sin\alpha\,d\alpha\,d\beta. \tag{46}$$

Variational integrals. The Hermitian character of $\mathscr{L}$ and $\mathscr{L}^2$ allows variational equivalents of Eqs. (7), (11) and (12). For example for Eq. (11) one obtains

$$(u,\mathscr{L}^2u) = (\mathscr{L}u,\mathscr{L}u) = \omega^2(u,u). \tag{47}$$

Substituting expansion (37) in Eq. (47) gives

$$\sum(\mathscr{L}u_{lk},\mathscr{L}u_{l'k'}) = \omega^2\sum(u_{lk},u_{l'k'}), \tag{48}$$

for specified (j,m,Q,P).

The sums in Eq. (48) are over l,k,l',k'. Integrations over angles can be carried out analytically, and the different matrix elements can be expressed as integrals over the magnitudes q and p. We postpone a more elaborate study of Eq. (48) and its numerical computations to another occasion. To gain some insight, however, we consider some of the simplest and the lowest modes in some details.

6. Modes belonging to $(j,m,P,Q) = (O,O,e,e)$

From Eq. (36) $j=0$ implies $l=k$. Requiring even parities further limits the choice to even integers. $l=k=0,2$, etc. Equation (35) for $j=m=0$ yields

$$|00ll\rangle = \sum_n Y_{ln}(\theta,\varphi)Y_{l,-n}(\alpha,\beta) < ln\,l,-n|00ll\rangle$$
$$= \frac{1}{4\pi}\sqrt{2l+1}\,P_l(\cos\Theta), \tag{49}$$

where we have used the addition theorem for Legendre polynomials and the Clebsch-Gordan coefficient $\langle ln\,l,-n|00ll\rangle = (-1)^{l+n}(2l+1)^{-1/2}$. See Brink and Satchler (1968, pp. 136–8). Equation (37) reduces to

$$u(\boldsymbol{q},\boldsymbol{p}) = \sum_l u_l(\boldsymbol{q},\boldsymbol{p}) = \frac{1}{4\pi}\sum_l\sqrt{2l+1}\,P_l(\cos\Theta)\bar{u}_l(q,p). \tag{50}$$

Letting $\mathscr{L}$ of Eq. (38) operate on $u(\boldsymbol{q},\boldsymbol{p})$ and reducing all angular integrals to single Legendre polynomials gives

$$\mathscr{L}u_l = \frac{1}{4\pi\sqrt{2l+1}}\{[\bar{\mathscr{L}}\bar{u}_l - l\bar{A}\bar{u}_l](l+1)P_{l+1} + [\bar{\mathscr{L}}\bar{u}_l + (l+1)\bar{A}\bar{u}_l]lP_{l-1}\}, \tag{51}$$

where

$$\bar{A} = -i\left[\frac{p}{q} - \frac{1}{p}\frac{dU}{dq}\right]. \tag{52}$$

A typical matrix element of Eq. (48) is $(\mathscr{L}u_{l'},\mathscr{L}u_l)$. After integrations over the angles one obtains

$$\begin{aligned}
(\mathscr{L}u_{l'},\mathscr{L}u_l) &= \frac{1}{(2l-1)(2l+3)}\{(2l^2+2l-1)(\bar{\mathscr{L}}\bar{u}_l,\bar{\mathscr{L}}\bar{u}_l) \\
&+ l(l+1)[(\bar{\mathscr{L}}u_l,\bar{A}\bar{u}_l) + (\bar{A}\bar{u}_l,\bar{\mathscr{L}}\bar{u}_l)] \\
&+ 2l^2(l+1)^2(\bar{A}\bar{u}_l,\bar{A}\bar{u}_l)\}\delta_{ll'} \\
&+ \frac{1}{\sqrt{(2l'+1)(2l+1)}}\left\{\frac{(l'+1)l}{2l-1}[(\bar{\mathscr{L}}\bar{u}_{l'},\bar{\mathscr{L}}\bar{u}_l)\right. \\
&+ (l+1)(\bar{\mathscr{L}}\bar{u}_{l'},\bar{A}\bar{u}_l) - l'(\bar{A}\bar{u}_{l'},\bar{\mathscr{L}}\bar{u}_l) \\
&- l'(l+1)(\bar{A}\bar{u}_{l'},\bar{A}\bar{u}_l)]\delta_{l',l-2} \\
&+ \frac{l'(l+1)}{2l'-1}[(\bar{\mathscr{L}}\bar{u}_{l'},\bar{\mathscr{L}}\bar{u}_l) \\
&- l(\bar{\mathscr{L}}\bar{u}_{l'},\bar{A}\bar{u}_l) + (l'+1)(\bar{A}\bar{u}_{l'},\bar{\mathscr{L}}\bar{u}_l) \\
&\left.- (l'+1)l(\bar{A}\bar{u}_{l'},\bar{A}\bar{u}_l)]\delta_{l'-2,l}\right\},
\end{aligned} \tag{53}$$

where all inner products on the right side are in the sense of Eq. (45). We note the coupling of the harmonic number l to $l\pm2$.

A typical matrix element on the right side of Eq. (48), after integrations over the angles is

$$(u_{l'},u_l) = (\bar{u}_{l'},\bar{u}_l)\delta_{l'l}. \tag{54}$$

There are no off diagonal terms in this matrix, for $|jmlk\rangle$'s of Eq. (35) constitute an orthonormal set.

Detailed solutions of Eqs. (48), (53), and (54) will be presented elsewhere. By way of illustration, however, we make a crude estimate of an eigenvalue. As a variational approximation we keep only $l=0$ term in Eq. (50). Equations (48), (53), and (54) give

$$\omega^2 = \frac{1}{3}\frac{(\bar{\mathscr{L}}\bar{u},\bar{\mathscr{L}}\bar{u})}{(\bar{u},\bar{u})}. \tag{55}$$

Let us assume a simple harmonic potential, $U=\frac{1}{2}\omega_0^2q^2$. This could be the potential of a self gravitating system of uniform density with $\omega_0^2 = \frac{4\pi}{3}G\rho$. Let $u(q,p)$ have the following trial value

$$\bar{u} = p\exp[-(p^2+\omega_0^2q^2)/2E_0], \tag{56}$$

where E_0 is a constant. The exponential term is a function of energy and is an integral of motion. It is incorporated in the trial function to make it square integrable. In fact $\bar{u}$ is the product of two simple harmonic wave functions, $\exp(-\omega_0^2 q^2/2E_0)$, the lowest wave function in q space, and $p\exp(-p^2/2E_0)$, the first excited function in p space. From Eq. (42) one has

$$\bar{\mathscr{L}}\bar{u} = \omega_0^2 q \exp[-(p^2+\omega_0^2 q^2)/2E_0]. \quad (57)$$

The integrals of Eq. (55) become

$$(\bar{\mathscr{L}}\bar{u}, \bar{\mathscr{L}}\bar{u}) = \frac{(2E_0)^4}{\omega_0} \int_0^\infty x^4 e^{-x^2} dx \int_0^\infty y^2 e^{-y^2} dy, \quad (58)$$

$$(\bar{u}, \bar{u}) = \frac{(2E_0)^4}{\omega_0^3} \int_0^\infty y^2 e^{-y^2} dy \int_0^\infty x^4 e^{-x^2} dx. \quad (59)$$

Substituting in Eq. (55) gives

$$\omega = \pm \frac{1}{\sqrt{3}} \omega_0 \quad (60)$$

As we shall see in the Appendix, the problem of simple harmonic potential is exactly solvable and the lowest non zero eigenvalue magnitudewise is ω_0. Our crudest calculation of the last few paragraphs is off by a factor of $1/\sqrt{3} = 0.6$. Note that the lowest eigenvalue of Liouville's equation is zero. The variational principle guarantees to produce eigenvalues larger than this which is true of Eq. (60).

7. Concluding remarks

The aim of the paper is to construct time-dependent solutions of Liouville's equation. For this purpose the symmetries of the Liouville operator are explored and its eigenfunctions are classified. An eigenfunction of $\mathscr{L}$ is a complex function of phase coordinates, whose real and imaginary parts are in turn eigenfunctions of $\mathscr{L}^2$. It is found that the five operators $\{\mathscr{L}^2, J^2, J_z, Q, P\}$ constitute a mutually commuting set. This enables one to search for the simultaneous eigenfunctions of the set and thus classify the eigenfunctions of $\mathscr{L}^2$ and $\mathscr{L}$ on the basis of eigenfunctions of $\{J^2, J_z, Q, P\}$. In the course of expansion of eigenfunctions of $\mathscr{L}^2$ in terms of those of the remaining operators the six dimensional phase space problem reduces to a two dimensional one in terms of the magnitudes of the position and momentum vectors. The two dimensional problem can then be put in differential or variational forms. Either way the computational simplification is enormous.

The potential entering Liouville's equation is assumed to be time-independent. Actual problems of theoretical astrophysics (let aside observational astronomy) are a good deal more complicated for this assumption to hold. The present paper lays the foundation on which the analysis of a more realistic problem will be built. To elucidate the point let us consider a self-gravitating spherically symmetric system. Let $F(E)$, E = energy, and $U(q)$ denote the equilibrium distribution and potential of the system. Let $\delta F = f(\boldsymbol{q}, \boldsymbol{p}, t)|dF/dE|^{1/2}$ and $\delta U(\boldsymbol{q}, t)$ be small perturbations on F and U, respectively. The time evolution of f is given by

$$i\frac{\partial f}{\partial t} = \mathscr{L}f - \mathrm{sign}(F_E)|F_E|^{1/2}\mathscr{L}\,\delta U, \quad (61)$$

$$\delta U = -G\int |F'_E|^{1/2} f(\boldsymbol{q}', \boldsymbol{p}')|\boldsymbol{q} - \boldsymbol{q}'|^{-1} d\boldsymbol{p}'\, d\boldsymbol{q}', \quad (62)$$

where $F_E = dF/dE$ (Sobouti, 1984). Equation (61) is an eigenvalue problem for f and the first order time variations of U are retained in it. Decomposing f into symmetric and antisymmetric terms in $\boldsymbol{p}$ and eliminating the symmetric term leads to an Antonov type of equation. Antonov's equation has all but the q symmetry of Eq. (11). There follows at once that all theorems, mode classes, and other characteristics that have been established so far (except those pertaining to the q symmetry) hold true for Antonov's problem. This linearised problem is treated in Paper III of this series.

As it stands the analysis can be used in several ways. 1) The eigenfunctions of Liouville's equation for some potential (no matter how idealized) can be used as a basis for expansion or as trial functions for iteration of the solutions of more complicated potentials. 2) In stellar systems with dimensions less than Jeans' wavelength contributions of δU to the eigenvalues of Eq. (61) is small compared with those of the remaining terms. In energy truncated distributions, this is shown to vary from 25% down to insignificant values (Sobouti, 1985). Omission of δU does not change the spectrum or the structure of the modes significantly. Thus, solutions proposed here may serve as reasonable approximations to solutions of Eq. (61).

In a series of papers (Sobouti, 1984, 85, 86) the author has studied the linearised Liouville equation. As an approximation he has assumed a perturbation on the distribution function whose odd p parity component has the form $f(\boldsymbol{q}, \boldsymbol{p}) = \boldsymbol{\xi}(\boldsymbol{q}) \cdot \boldsymbol{p}$, where $\boldsymbol{\xi}(\boldsymbol{q})$ is a vector in $\boldsymbol{q}$ space and in turn is expanded in terms of vector spherical harmonics. The referee of 1986 paper expressed reservations on this ansatz and recommended a further justification. The present analysis provides this justification. The angular part of $\boldsymbol{\xi} \cdot \boldsymbol{p}$ turns out to be a member of $|jmlk\rangle$ functions of Eq. (35). In case $\boldsymbol{\xi} \cdot \boldsymbol{p} = f(q)\boldsymbol{q} \cdot \boldsymbol{p} = f(q)qp\cos\Theta$, the subject of 1984 and 1985 papers, $\cos\Theta$ is a member of $|00ll\rangle$ of Eq. (49). Thus, the expression $\boldsymbol{\xi}(\boldsymbol{q}) \cdot \boldsymbol{p}$ is a term from the expansions of Eqs. (37) and (49) and is a legitimate variational approximation. Actually in the case of simple harmonic potential a term such as $\boldsymbol{q} \cdot \boldsymbol{p}$ is an exact eigenfunction of $\mathscr{L}^2$. The author finds it gratifying that some members of his earlier ansatz are indeed exact solutions of simple harmonic potentials, the potential of self-gravitating systems of uniform density.

Acknowledgements. I am grateful to professor Frank H. Shu for illuminating and stimulating comments. The research reported here was supported in part by The Research Council of Shiraz University.

Appendix: simple harmonic potential

The Liouville equation is exactly solvable in this case and much can be learned from it as regards the analytical properties and actual computations of the more general problems. We include this brief expose here to elucidate the concepts developed in the text and to prove the completeness of the set of the eigenfunctions.

In Cartesian coordinates let $U = \frac{1}{2}\omega_0^2(x_1^2 + x_2^2 + x_3^2)$ be the potential. The Liouville operator is

$$\mathscr{L} = -i\left(p_i \frac{\partial}{\partial x_i} - \omega_0^2 x_i \frac{\partial}{\partial p_i}\right). \quad (A.1)$$

Let the maximum energy available to the oscillator be E_0. This limits the accessible region of the phase space to inside the sphere $p^2 + \omega_0^2 r^2 \le 2E_0$. One may easily verify that the following is an eigenfunction of $\mathscr{L}$.

$$\begin{aligned} f_i &= p_i + i\omega_0 x_i \quad &&\text{inside } E, \\ &= 0 &&\text{outside } E_0. \end{aligned} \quad (A.2)$$

Propositions of Theorem 2.2, 2.3, and 2.4 are seen in the following relations:

$$\mathscr{L}f_i = \omega_0 f_i\,, \tag{A.3}$$

$$\mathscr{L}f_i^* = -\omega_0 f_i^*\,, \tag{A.4}$$

$$\mathscr{L}f_i^* f_i = 0\,, \quad \sum_i f_i^* f_i = \tfrac{1}{2}(p^2 + \omega_0^2 r^2) = \text{energy}\,, \tag{A.5}$$

$$\mathscr{L}(f_i f_j^* - f_i^* f_j) = 0\,,$$

$$(f_i f_j^* - f_i^* f_j) = k\text{th. component of angular momentum}\,, \tag{A.6}$$

$$\mathscr{L}f_i^n = n\omega_0 f_i^n\,, \quad \mathscr{L}f_i^{*n} = -n\omega_0 f_i^{*n}\,. \tag{A.7}$$

Eigenfunctions integrate to zero (Theorem 2.4)

$$\int f_i\, d\boldsymbol{x}\, d\boldsymbol{p} = 0\,, \tag{A.8}$$

for f_i is odd in x and p, and the domain of integration is symmetric about the origin of the phase coordinates.

Completeness of the set $\{f_i^n, f_i^{*n}\}$, $n = 0, 1, 2, \ldots, i = 1, 2, 3$

The set $\{x_i^n\}$, $n = 0, 1, \ldots, i = 1, 2, 3$, is complete in the finite interval $0 \le x_i \le a_i$. So is the set $\{p_i^n\}$ in the interval $0 \le p_i \le b_i$. One, however, has

$$x_i = \frac{1}{2i\omega_0}(f_i - f_i^*)\,, \tag{A.9}$$

$$p_i = \tfrac{1}{2}(f_i + f_i^*)\,. \tag{A.10}$$

There follows that $\{x_i^n, p_i^n\}$ is equivalent to $\{f_i^n, f_i^{*n}\}$ and the latter set is also complete. QED.

References

Antonov, V.A.: 1960, *Ast. Zh.* **37**, 918; 1961, *Sov. Astron.* **4**, 859

Balescu, R.: 1975, *Equilibrium and non-equilibrium statistical mechanics*, Wiley Interscience, New York

Barnes, J., Goodman, J., Hut, P.: 1986, *Astrophys. J.*, **300**, 112

Brink, D.M., Sactchler, G.K.: 1968, *Angular momentum*, Clarendon Press, Oxford

Condon, E.U., Shortley, G.H.: 1935, *Theory of atomic spectra*, Univ. Press, Cambridge

George, C., Prigogine, I.: 1979, *Physica* **99***A*, 369

Lin, C.C., Yuan, C., Shu, F.H.: 1969, *Astrophys. J.*, **155**, 721

Lynden-Bell, D.: 1962, *Monthly Notices Roy. Astron. Soc.*, **124**, 279

Lynden-Bell, D.: 1967, in *Relativity and astrophysics, 2. Galactic structure*, pp. 131–168, ed., Ehlers, J., Am. Math. Soc.

Prigogine, I.: 1962, *Non-equilibrium statistical mechanics*, Wiley Interscience, New York

Prigogine, I.: 1980, *From being to becoming*, Freeman and Co., New York

Rose, M.E.: 1957, *Elementary theory of angular momentum*, J. Wiley, New York

Rotenberg, M., Bivins, R., Metropolis, N., Wooten JR. J.K.: 1959, *The 3-j and 6-j symbols*, Technology Press, Cambridge, Mass.

Schonberg, M.: 1952, *Nuevo Cimento* **9**, 1139

Shu, F.M.: 1970, *Astrophys. J.*, **160**, 89

Sobouti, Y.: 1984, *Astron. Astrophys.*, **140**, 82

Sobouti, Y.: 1985, *Astron. Astrophys.*, **147**, 61

Sobouti, Y.: 1986, *Astron. Astrophys.*, **169**, 95