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ASTRONOMY AND ASTROPHYSICS

Liouville's equation

IV. The full symmetries of quadratic potentials*

Y. Sobouti[1,2] and M.H. Dehghani[1,2]

[1] Department of Physics and Biruni Observatory, Shiraz University, Shiraz 71454, Iran**
[2] Center for Theoretical Physics and Mathematics AEOI, Tehran, Iran

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Abstract. A systematic study of the symmetries of Liouville's equation for an arbitrary potential is presented. The method is applied to the case of quadratic potentials. The symmetry group of the latter turns out to be $GL(3,c)$ with the noncompact subgroup $SL(3,c)$. The latter, in turn, has the subgroups $SU(3)$, and $SO(3)$, $SO(3,1)$ and $SU(2,1)$ of which the first two are compact and the last two noncompact. Finally, the largest compact and the largest noncompact subgroups of $GL(3,c)$ are used to classify the eigenmodes of Liouville's equation for quadratic potentials

Key words: galaxies – dynamics and evolution – Liouville's equation – symmetries and normal modes

1. Introduction

It is an assumption of traditional stellar dynamics that Liouville's equation governs the time evolution of stellar systems. An inevitable consequence of such a premise is that (a) at least some modes of instability of stellar systems may be those of Liouville's equation; and (b) stellar systems might undergo periodic changes of definite patterns in configuration and velocity spaces. For, Liouville's equation exhibits eigenmodes of oscillation. While it is not feasible to observe the astronomically long periods of oscillation, the patterns of changes, i.e. the eigenfunctions, may be amenable to observation by analyzing the CCD records of brightness and velocity distributions on the visible disks of galaxies and globular clusters. This is our motivation for scrutinizing Liouville's equation, if not for its own merits. The work is a continuation of a series of papers on Liouville's equation (Sobouti 1989a, b; Sobouti & Samimi 1989; hereafter Papers I, II and III, respectively). In this paper we introduce a systematic method for finding the symmetries of Liouville's equation for an arbitrary potential, in general, and for quadratic potentials in some detail.

It is well known (Jauch & Hill 1940) that the Hamiltonian of the three-dimensional harmonic oscillator is invariant under $SU(3)$. In the quantum-mechanical context, this is the minimal group that completely spans the state of individual degenerate levels of the harmonic oscillator, and is often called the degeneracy group. A larger dynamical group which contains a set of operators that determine the transition probabilities between states have also been investigated (see for example Hwa & Nuyts 1966 and Haskell & Wybourne 1973). In Sect. 2 we write down an algorithm for a systematic study of the symmetries of Liouville's equation for an arbitrary potential. In Sect. 3 we find $GL(3,c)$ as the symmetry group of Liouville's equation for quadratic potentials. Among the notable subgroups of $GL(3,c)$ are the compact subgroups $SO(3)$ and $SU(3)$, and the noncompact ones $SO(3,1)$ and $SU(2,1)$. These are discussed in Sect. 4. Finally, in Sect. 5 we introduce a complete set of mutually commuting operators. They enable one to classify the eigenfunctions of Liouville's operator into invariant subspaces.

2. Symmetry transformations

Let $(\boldsymbol{q},\boldsymbol{p})$ denote the collection of configuration and momentum coordinates specifying a dynamical system. Let $\mathscr{H}$ be the Hilbert space of complex-valued functions $f(\boldsymbol{q},\boldsymbol{p})$ in which the inner product is defined as

$$(f,g)=\int f^* g\,\mathrm{d}\boldsymbol{q}\,\mathrm{d}\boldsymbol{p}<\infty,\quad f,g\in\mathscr{H}. \tag{1}$$

Liouville's equation on $\mathscr{H}$ may be written as

$$\mathrm{i}\frac{\partial f}{\partial t}=\mathscr{L}f=\mathrm{i}[H,f]_{\mathrm{P}},\quad \mathscr{L}=-\mathrm{i}\left(p_i\frac{\partial}{\partial q_i}-\frac{\partial U}{\partial q_i}\frac{\partial}{\partial p_i}\right), \tag{2}$$

where $H=p^2/2m+U(\boldsymbol{q})$ is the Hamiltonian, $[\ldots,\ \ldots]_{\mathrm{P}}$ is the Poisson bracket, and $\mathscr{L}$ is Liouville's operator. The reason for including i in Eq. (2) is to render $\mathscr{L}$ Hermitian. That is, $(\mathscr{L}f,g)=(f,\mathscr{L}g); f,g\in\mathscr{H}$. See Prigogine (1962) and Paper I for more details. For a systematic study of the symmetries of Eq. (2) we follow a procedure parallel to that of Killing to obtain the isometries of curved spacetimes. We look for those infinitesimal transformations of $(\boldsymbol{q},\boldsymbol{p})$ to $(\boldsymbol{q}',\boldsymbol{p}')$ that leave $\mathscr{L}$ form-invariant. Hence, let

$$q_i'=q_i+\varepsilon\xi_i(\boldsymbol{q},\boldsymbol{p}), \tag{3a}$$

$$p_i'=p_i+\varepsilon\eta_i(\boldsymbol{q},\boldsymbol{p}), \tag{3b}$$

Send offprint requests to: M.H. Dehghani (first address above)
* Contribution No. 23 Biruni Observatory
** Permanent address

where ε is an infinitesimal parameter. It is a matter of straightforward calculation to show that $\mathscr{L}$ transforms as follows:

$$\mathscr{L}(\boldsymbol{q}',\boldsymbol{p}')=\mathscr{L}(\boldsymbol{q},\boldsymbol{p})+\varepsilon(\mathscr{L}\xi_i+\mathrm{i}\eta_i)\frac{\partial}{\partial q_i}-\varepsilon\left(\mathscr{L}\eta_i-\mathrm{i}\xi_j\frac{\partial^2 U}{\partial q_j\partial q_i}\right)\frac{\partial}{\partial p_i}, \tag{4}$$

where $\mathscr{L}(\boldsymbol{q}',\boldsymbol{p}')$ is the same Liouville's operator in $(\boldsymbol{q}',\boldsymbol{p}')$ coordinates. Form-invariance of $\mathscr{L}$ for all $f\in\mathscr{H}$ leads to

$$\mathscr{L}\xi_i+\mathrm{i}\eta_i=0, \tag{5a}$$

$$\mathscr{L}\eta_i-\mathrm{i}\xi_j\frac{\partial^2 U}{\partial q_j\partial q_i}=0. \tag{5b}$$

Upon elimination of η, one obtains

$$\mathscr{L}^2\xi_i-\xi_j\frac{\partial^2 U}{\partial q_j\partial q_i}=0. \tag{5c}$$

As a consequence of these transformations, a phase space function changes into $f(\boldsymbol{q}',\boldsymbol{p}')=f(\boldsymbol{q},\boldsymbol{p})+\varepsilon\chi f(\boldsymbol{q},\boldsymbol{p});f\in\mathscr{H}$, where the generator χ is given by

$$\chi=\xi_i\frac{\partial}{\partial q_i}+\eta_i\frac{\partial}{\partial p_i}. \tag{6}$$

One may readily verify that

$$[\mathscr{L},\chi]=0, \tag{7}$$

where $[\ldots,\ldots]$ is a commutator bracket. So much for generalities. Further progress requires specific assumptions with regard to potential $U(\boldsymbol{q})$. It is shown in Papers I and II that for spherically symmetric potentials there are three generators that obey the angular momentum algebra and it is concluded that the symmetry group of $\mathscr{L}$ is $SO(3)$. Some more symmetry operators for quadratic potentials are also given there, but not all. Here, we study the latter case in detail and provide the full symmetry group. The symmetries of Liouville's equation for r^{-1} potential will be presented elsewhere.

3. Maximally symmetric quadratic potentials

The case to be studied, as any aspect of quadratic potentials is exactly and analytically solvable. It unravels many of the idiosyncrasies of questions and answers that arise in the course of analysis. Apart from their academic merits, however, quadratic potentials do find important applications. The central regions of extended stellar systems, the de Sitter and anti de Sitter space-times are examples. Furthermore, the complete eigensolutions of Liouville's equation for quadratic potentials can be used either as a basis for the function space $\mathscr{H}$ or as approximate solutions for less-symmetric potentials.

For $U=\frac{1}{2}q_iq_i$, solutions of Eqs. (5a)–(5c) are

$$\xi_i=a_{ij}q_j+b_{ij}p_j, \tag{8a}$$

$$\eta_i=-b_{ij}q_j+a_{ij}p_j, \tag{8b}$$

where a_{ij} and b_{ij} are eighteen real constants. The infinitesimal transformations induced by Eqs. (3a), (3b) and (8) are continuous, linear, one to one and invertible. Thus, they constitute an eighteen-parameter Lie group. At a first glance this is a general linear group in a 3-dimensional complex space $GL(3,c)$. For, one may combine Eqs. (3a), (3b) and (8) into the following form:

$$z_i'=(\delta_{ij}+\varepsilon c_{ij})z_j, \tag{9}$$

where $z_i=p_i+\mathrm{i}q_i$ and $c_{ij}=a_{ij}+\mathrm{i}b_{ij}$. As it is well known and it will also be discussed below $GL(3,c)$ is neither simple nor semisimple. But it has a sixteen parameter semisimple subgroup, $SL(3,c)$. One expects it to contain the invariance group of the Hamiltonian as a subgroup. For, if the Hamiltonian remains form-invariant the Poisson bracket, $[H,f]_{\mathrm{P}}$, must also do so. We will return to this point in Sect. 4.2.2.

The generators of Eq. (6) for the transformations of Eqs. (8a) and (8b) become

$$\chi_\alpha=-\mathrm{i}\left\{a_{jk}\left(q_j\frac{\partial}{\partial q_k}+p_j\frac{\partial}{\partial p_k}\right)+b_{jk}\left(p_j\frac{\partial}{\partial q_k}-q_j\frac{\partial}{\partial p_k}\right)\right\}, \tag{10}$$

where $\alpha=1,2,\ldots,18$ correspond to eighteen independent choices of a_{ij} and b_{ij}. Naturally, χ_α's obey a Lie algebra. By straightforward calculations, one finds

$$[\chi_\alpha,\chi_\beta]=\mathrm{i}C^\gamma_{\alpha\beta}\chi_\gamma,\quad \alpha,\beta,\gamma=1,2,\ldots,18, \tag{11a}$$

where $C^\gamma_{\alpha\beta}$'s are real structure constants and satisfy the Jacobi identity,

$$C^\delta_{\alpha\beta}C^\rho_{\gamma\delta}+C^\delta_{\beta\gamma}C^\rho_{\alpha\delta}+C^\delta_{\gamma\alpha}C^\rho_{\beta\delta}=0. \tag{11b}$$

Like the group itself this algebra is neither simple nor semisimple.

4. Classification of symmetries

For each of the 3×3 matrices a_{jk} and b_{jk} we choose one unit matrix, three antisymmetric matrices, and five traceless symmetric matrices. This allows the following identification of the subgroups.

4.1. *Abelian invariant subgroups*

There are two such subgroups:

(a) For $a_{ij}=0$ and $b_{ij}=\delta_{ij}$, the generator χ_α turns out to be Liouville's operator $\mathscr{L}$ itself. The corresponding coordinate transformations are

$$q_i'=q_i+\varepsilon p_i,\quad p_i'=p_i-\varepsilon q_i. \tag{12}$$

Here, the mass of the oscillator and the force constant are set equal to one. This gives $\dot{q}_i=p_i$ and $\dot{p}_i=-\partial U/\partial q_i=-q_i$. Hence, the transformations of Eq. (12) express the time evolution of the system along a phase trajectory during the time interval ε. They may also be interpreted as rotations in q_i–p_i planes through the infinitesimal angle ε. The transformation of Eq. (12) form an Abelian invariant subgroup of $GL(3,c)$. For, its generator $\mathscr{L}$ commutes with all other χ_α's.

(b) For $a_{ij}=\delta_{ij}$ and $b_{ij}=0$, the generator χ_α is

$$F=-\mathrm{i}\left(p_i\frac{\partial}{\partial p_i}+q_i\frac{\partial}{\partial q_i}\right). \tag{13}$$

The corresponding transformation is

$$q_i'=q_i+\varepsilon q_i,\quad p_i'=p_i+\varepsilon p_i. \tag{14}$$

This is a scale transformation with Jacobian$=1+6\varepsilon\neq1$. This transformation is the only one which does not preserve the volume of phase space. The operator F commutes with $\mathscr{L}$ and with the remaining sixteen χ_α's. Hence, the scale transformation also constitutes an Abelian invariant subgroup of $GL(3,c)$. The two together form an ideal for the Lie algebra. Hence, the Lie algebra and its associated group are neither simple nor semi-simple.

4.2. *The compact subgroups*

4.2.1. Rotation invariance subgroup; SO(3)

The three transformations for which $a_{jk}=\varepsilon_{ijk}$ (Levi Civita symbols), $i=1, 2, 3$ and $b_{jk}=0$ are

$$q'_j=q_j+\varepsilon q_k, \quad q'_k=q_k-\varepsilon q_j, \tag{15a}$$

$$p'_j=p_j+\varepsilon p_k, \quad p'_k=p_k-\varepsilon p_j. \tag{15b}$$

There are three sets of Eqs. (15a) and (15b) corresponding to the pairs $(j, k)=(1, 2), (2, 3)$ and $(3, 1)$. These are simultaneous rotations in $(q_j\text{–}q_k)$ and $(p_j\text{–}p_k)$ planes. They leave q^2 and p^2 and, consequently, both the Lagrangian, $\frac{1}{2}(p^2-q^2)$, and the Hamiltonian, $\frac{1}{2}(p^2+q^2)$, invariant. The three generators are

$$J_i=J_i^\dagger=-\mathrm{i}\varepsilon_{ijk}\left(p_j\frac{\partial}{\partial p_k}+q_j\frac{\partial}{\partial q_k}\right), \tag{15c}$$

with the angular momentum algebra

$$[J_i, J_j]=\mathrm{i}\varepsilon_{ijk}J_k. \tag{16}$$

The corresponding Casimir operator, i.e. the operator that commutes with all members of the algebra, is J^2. Sobouti (1989a, b) has shown that this $SO(3)$ symmetry holds true for any spherically symmetric potential. Sobouti & Samimi (1989) have demonstrated the same for the linearized Liouville equation of the spherically symmetric stellar systems and utilized in the computations of normal modes of oscillations.

4.2.2. Invariance subgroup of Hamiltonian; SU(3)

This group has been considered by many authors and mainly for quantum-mechanical and field-theoretic purposes. Notable are the investigations of Jauch & Hill (1940), Fradkin (1964), Barut (1965) and Hwa & Nuyts (1966). Invariance of the Hamiltonian, $H=\frac{1}{2}(p^2+q^2)$, is realized by anti-symmetric a_{jk} and symmetric b_{jk}'s. The former is already discussed. The latter, the symmetric b_{jk}, can be chosen as one identity matrix and five traceless symmetric matrices. The identity, $b_{jk}=\delta_{jk}$, was discussed in Sect. 4.1 (a). Of the five symmetric matrices three are chosen as $b_{jk}=b_{kj}=|\varepsilon_{ijk}|$; the transformations are

$$q'_j=q_j+\varepsilon p_k, \quad p'_k=p_k-\varepsilon q_j, \tag{17a}$$

three sets for $(j, k)=(1, 2), (2, 3)$ and $(3, 1)$. These are rotations in $(q_j\text{–}p_k)$ planes. The corresponding generators χ_α are

$$L_i=L_i^\dagger=-\mathrm{i}|\varepsilon_{ijk}|\left(p_j\frac{\partial}{\partial q_k}-q_j\frac{\partial}{\partial p_k}\right). \tag{17b}$$

A fourth symmetric traceless matrix can be chosen the diagonal, $b_{11}=-b_{22}=1$ and $b_{33}=0$. The transformation and its generator are

$$q'_1=q_1+\varepsilon p_1, \quad p'_1=p_1-\varepsilon q_1, \tag{18a}$$

$$q'_2=q_2-\varepsilon p_2, \quad p'_2=p_2+\varepsilon q_2, \tag{18b}$$

$$M=M^\dagger=-\mathrm{i}\left(p_1\frac{\partial}{\partial q_1}-p_2\frac{\partial}{\partial q_2}-q_1\frac{\partial}{\partial p_1}+q_2\frac{\partial}{\partial p_2}\right). \tag{18c}$$

Finally, for the last of the traceless symmetric b_{ij}, we choose $b_{11}=b_{22}=\frac{1}{2}b_{33}=\frac{1}{3}$, and all other elements zero. The corresponding transformations are similar to those of Eqs. (18a) and (18b). Its generator is

$$Y=Y^\dagger=-\frac{\mathrm{i}}{3}\left(p_1\frac{\partial}{\partial q_1}+p_2\frac{\partial}{\partial q_2}-2p_3\frac{\partial}{\partial q_3}\right.$$
$$\left.-q_1\frac{\partial}{\partial p_1}-q_2\frac{\partial}{\partial p_2}+2q_3\frac{\partial}{\partial p_3}\right). \tag{19}$$

The transformations of Eqs. (18a)–(18c) and (19) may also be understood as rotations in real phase space, or as time evolutions forward and backward in time. See the interpretation of Eq. (12). The set $\{J_i, L_i, M, Y\}$ has the $SU(3)$ algebra, expressed in Eq. (16) and below

$$[L_i, L_j]=-\mathrm{i}\varepsilon_{ijk}J_k, \quad [M, Y]=0, \tag{20a}$$

$$[J_1, L_1]=\mathrm{i}(M-3Y), \quad [J_2, L_2]=\mathrm{i}(M+3Y),$$
$$[J_3, L_3]=-2\mathrm{i}M, \tag{20b}$$

$$[J_1, L_2]=-\mathrm{i}L_3, \quad [J_2, L_3]=-\mathrm{i}L_1, \quad [J_3, L_1]=-\mathrm{i}L_2, \tag{20c}$$

$$[J_1, M]=-\mathrm{i}L_1, \quad [J_2, M]=-\mathrm{i}L_2, \quad [J_3, M]=+2\mathrm{i}L_3, \tag{20d}$$

$$[J_1, Y]=+\mathrm{i}L_1, \quad [J_2, Y]=-\mathrm{i}L_2, \quad [J_3, Y_1]=0, \tag{20e}$$

$$[L_1, M]=+\mathrm{i}J_1, \quad [L_2, M]=+\mathrm{i}J_2,$$
$$[L_3, M]=-2\mathrm{i}J_3, \tag{20f}$$

$$[L_1, Y]=-\mathrm{i}J_1, \quad [L_2, Y]=+\mathrm{i}J_2, \quad [L_3, Y]=0. \tag{20g}$$

The quadratic Casimir operator for this algebra is

$$\mathscr{K}=(J^2+L^2+M^2+3Y^2). \tag{21}$$

4.3. *Noncompact subgroups*

4.3.1. Invariance subgroup of Lagrangian, the Lorentz group SO(3, 1)

The transformations for $b_{jk}=\varepsilon_{ijk}$ and $a_{jk}=0$ are

$$q'_j=q_j+\varepsilon p_k, \quad p'_k=p_k+\varepsilon q_j, \tag{22a}$$

three sets for $(j, k)=(1, 2), (2, 3)$ and $(3, 1)$. These are boosts, similar to those of Lorentz', in the $(q_j\text{–}p_k)$ plane. As is known they are not unitary transformations. The parameter ε, if interpreted as imaginary rotation angle, does not have a finite range. These transformations together with those of Eqs. (15a)–(15c) leave the Lagrangian form-invariant. The corresponding generators in addition to J_i's are

$$I_i=\mathrm{i}\varepsilon_{ijk}\left(p_j\frac{\partial}{\partial q_k}-q_j\frac{\partial}{\partial p_k}\right)\neq I^\dagger. \tag{22b}$$

The generators are not Hermitian. For, the underlying coordinate transformations of Eq. (21) are not orthogonal. One readily verifies that the set $\{J_i, I_i\}$ has the closed Lie algebra of $SO(3, 1)$. That is, in addition to Eq. (16),

$$[J_i, I_j]=\mathrm{i}\varepsilon_{ijk}I_k, \tag{23a}$$

$$[I_i, I_j]=-\mathrm{i}\varepsilon_{ijk}J_k. \tag{23b}$$

The two Casimir operator of this subgroup are J^2-I^2 and $\boldsymbol{J}\cdot\boldsymbol{I}$.

4.3.2. A further subgroup; SU(2, 1)

The symmetries belonging to the five traceless symmetric a_{jk}'s remain to be discussed. Three of these are induced by $a_{jk}=a_{kj}=|\varepsilon_{ijk}|$, and $b_{jk}=0$, the transformations are

$$q'_j=q_j+\varepsilon q_k, \quad q'_k=q_k+\varepsilon q_j, \tag{24a}$$

$$p'_j=p_j+\varepsilon p_k, \quad p'_k=p_k+\varepsilon q_j, \tag{24b}$$

three sets for $(j, k)=(1, 2), (2, 3)$ and $(3, 1)$. As in Eqs. (22a) and (22b), these transformations cannot be interpreted as rotations through real angles. They, too, resemble Lorentz boosts, and are not orthogonal. The bilinear forms kept invariant under these transformations are the three expressions $H_j-H_k=\frac{1}{2}[p_j^2+q_j^2-(p_k^2+q_k^2)]$, $(j, k)=(1, 2), (2, 3)$ and $(3, 1)$. The corresponding generators are

$$K_i=-\mathrm{i}|\varepsilon_{ijk}|\left(p_j\frac{\partial}{\partial p_k}+q_j\frac{\partial}{\partial q_k}\right). \tag{24c}$$

The fourth transformation induced by $a_{11}=-a_{22}=1$ is a scale expansion in q_1-, p_1-directions and a scale contraction by the same factor in q_2-, p_2-directions. This is area-preserving but not orthogonal. The transformation and its generator, are

$$q'_1=q_1+\varepsilon q_1, \quad p'_1=p_1+\varepsilon p_1, \tag{25a}$$

$$q'_2=q_2-\varepsilon q_2, \quad p'_2=p_2-\varepsilon p_2, \tag{25b}$$

$$R=-\mathrm{i}\left(p_1\frac{\partial}{\partial p_1}-p_2\frac{\partial}{\partial p_2}+q_1\frac{\partial}{\partial q_1}-q_2\frac{\partial}{\partial q_2}\right). \tag{25c}$$

Finally, for the last of the traceless symmetric a_{ij}'s, let $a_{11}=a_{22}=-\frac{1}{2}a_{33}=\frac{1}{3}$, and all other elements zero. These are scale expansions in 1- and 2-directions and a scale contraction by twice as much in 3-direction. The corresponding generator is

$$S=-\frac{\mathrm{i}}{3}\left(p_1\frac{\partial}{\partial p_1}+p_2\frac{\partial}{\partial p_2}-2p_3\frac{\partial}{\partial p_3}\right.$$
$$\left.+q_1\frac{\partial}{\partial q_1}+q_2\frac{\partial}{\partial q_2}-2q_3\frac{\partial}{\partial q_3}\right). \tag{26}$$

The algebra of the set $\{J_i, K_i, R, S\}$ is closed and isomorphic to $SU(2, 1)$. This is given in Eq. (16) and below.

$$[K_i, K_j]=+\mathrm{i}\varepsilon_{ijk}J_k, \quad [R, S]=0, \tag{27a}$$

$$[J_1, K_1]=\mathrm{i}(R-3S), \quad [J_2, K_2]=\mathrm{i}(R+3S),$$
$$[J_3, K_3]=-2\mathrm{i}R, \tag{27b}$$

$$[J_1, K_2]=-\mathrm{i}K_3, \quad [J_2, K_3]=-\mathrm{i}K_1, \quad [J_3, K_1]=-\mathrm{i}K_2, \tag{27c}$$

$$[J_1, R]=-\mathrm{i}K_1, \quad [J_2, R]=-\mathrm{i}K_2, \quad [J_3, R]=+2\mathrm{i}K_3, \tag{27d}$$

$$[J_1, S]=+\mathrm{i}K_1, \quad [J_2, S]=-\mathrm{i}K_1, \quad [J_3, S]=0, \tag{27e}$$

$$[K_1, R]=-\mathrm{i}J_1, \quad [K_2, R]=-\mathrm{i}J_2, \quad [K_3, R]=+2\mathrm{i}J_3, \tag{27f}$$

$$[K_1, S]=+\mathrm{i}J_1, \quad [K_2, S]=-\mathrm{i}J_2, \quad [K_3, S]=0. \tag{27g}$$

The quadratic Casimir operator for this subalgebra is $J^2-K^2-R^2-3S^2$.

Let us close this section by summarizing that (1) all orthogonal transformations of Sects. 4.1(a) and 4.2 are canonical, constitute compact subgroups and lead to Hermitian generators, and (2) The nonorthogonal transformations of Sects 4.1(b) and 4.3 are not canonical, constitute noncompact subgroups and lead to non-Hermitian generators.

5. Normal modes

The eigenvalue equation $\mathscr{L} f_n=n f_n$ has a highly degenerate spectrum. For, any f_n multiplied by an arbitrary function of the constants of motion (energy, angular momentum, etc.) is again an eigenfunction belonging to the same n. To unravel the degeneracy we must introduce along with $\mathscr{L}$, a set of mutually commuting operators, and seek their simultaneous eigensolutions. The following two alternatives are proposed.

5.1. Normal modes through SU(3)

The largest compact subgroup of symmetries of $\mathscr{L}$ is $SU(3)$. It is of rank 2. That is, it has two mutually commuting and two independent Casimir operators at most. For a commuting set of four we choose $\{\mathscr{L}, \mathscr{K}, \frac{1}{2}M, Y\}$. The $SU(3)$ group has been studied extensively in other branches of physics and notably in connection with elementary particles. In its application to the present problem we will borrow notations and terminologies from the latter discipline. See, for example, Greiner & Muller (1989). The smallest representations of $SU(3)$ are a triplet and an antitriplet of modes. Here, they could be chosen as follows

$$[3]:\{f_i=p_i+\mathrm{i}q_i;\ i=1, 2, 3\}, \tag{28a}$$

$$[\bar{3}]:\{\bar{f}_i=f_i^*\}. \tag{28b}$$

As they are, f_i and $\bar{f}_i$ are not square-integrable. However, multiplied by e^{-E}, $E=\frac{1}{2}(p^2+q^2)$, they become so and become a member of the Hilbert space $\mathscr{H}$. Nonetheless, we will not include e^{-E} explicitly as part of the modes. Instead we will modify the inner product of Eq. (1) to read as

$$(f, g)=\int f^* g \mathrm{e}^{-2E}\,\mathrm{d}\boldsymbol{q}\,\mathrm{d}\boldsymbol{p}. \tag{29}$$

With this inner product f_i and $\bar{f}_i$ requires a factor $(2\pi)^{-1/2}$ to become normalized. We will not, however, attend to this question at this stage. The set of the eigenvalues, (n, k, m, y) of the operators $\{\mathscr{L}, \mathscr{K}, \frac{1}{2}M, Y\}$ for different modes may be read from the following equations

$$\mathscr{L} f_i=f_i, \quad \mathscr{L}\bar{f}_i=-\bar{f}_i, \tag{30a}$$

$$\mathscr{K} f_i=\tfrac{16}{3}f_i, \quad \mathscr{K}\bar{f}_i=\tfrac{16}{3}\bar{f}_i, \tag{30b}$$

$$\tfrac{1}{2}M f_i=\tfrac{1}{2}f_1, \quad \tfrac{1}{2}M f_2=-\tfrac{1}{2}f_2, \quad \tfrac{1}{2}M f_3=0, \tag{30c}$$

$$\tfrac{1}{2}M\bar{f}_1=-\tfrac{1}{2}\bar{f}_1, \quad \tfrac{1}{2}M\bar{f}_2=+\tfrac{1}{2}\bar{f}_2, \quad \tfrac{1}{2}M\bar{f}_3=0, \tag{30d}$$

$$Yf_1=\tfrac{1}{3}f_1, \quad Yf_2=-\tfrac{1}{3}f_2, \quad Yf_3=-\tfrac{2}{3}f_3, \tag{30e}$$

$$Y\bar{f}_1=-\tfrac{1}{3}\bar{f}_1, \quad Y\bar{f}_2=+\tfrac{1}{3}\bar{f}_2, \quad Y\bar{f}_3=+\tfrac{2}{3}\bar{f}_3. \tag{30f}$$

Figure 1 depicts the modes in a $\frac{1}{2}m$–y diagram. The two triplets are the exact analogs of the quark and antiquark triplets in elementary particles. The eigenvalues m and y play the role of the isospin and hypercharge, respectively. Larger representation of the group, that is the higher-order modes, may be constructed by the direct products of the triplets and antitriplets. The procedure could be the same as the construction of composite particles from quarks and antiquarks. Two examples are given below.

(1) The direct product of a triplet and an antitriplet is the

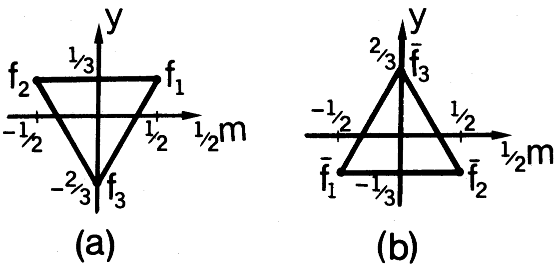

Fig. 1a and b. $1/2m$–y diagram of (a) triplet and (b) antitriplet modes. See Eqs. (26) and (30)

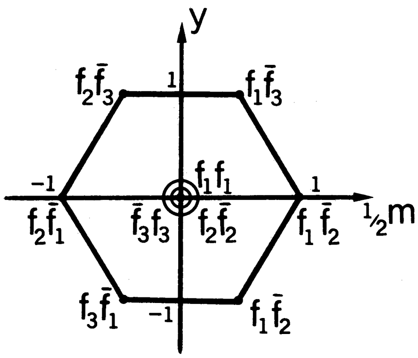

Fig. 2. $1/2m$–y diagram of $[3] \otimes [\bar{3}]$ multiplet. See Eq. (31), and comments thereof

direct sum of a sextet and a triplet as follows:

$$[3] \otimes [\bar{3}] = [6] \oplus [3] : \{f_1 \bar{f}_2, f_3 \bar{f}_2, f_3 \bar{f}_1, f_2 \bar{f}_1, f_2\ \bar{f}_3, f_1 \bar{f}_3\} \oplus \{f_1 \bar{f}_1, f_2 \bar{f}_2, f_3 \bar{f}_3\}. \tag{31}$$

In Fig. 2, the sextet modes occupy the corners of the hexagon. The remaining three are at the center. Thus, the center is threefold degenerate. To remove this degeneracy one might invite in one or more operators from the noncompact group $SU(2, 1)$. For example the R operator of Eq. (25c) gives $\mathrm{i}R f_1 \bar{f}_1 = f_1 \bar{f}_1$, $\mathrm{i}R f_2 \bar{f}_2 = -f_2 \bar{f}_2$ and $\mathrm{i}R f_3 \bar{f}_3 = 0$. The effect of $\mathscr{L}$ and $\mathscr{K}$ on these sextet and triplet is

$$\mathscr{L}\{[3] \otimes [\bar{3}]\} = 0, \tag{32a}$$

$$\mathscr{K}[3] = 8[3], \tag{32b}$$

$$\mathscr{K}[6] = 12[6]. \tag{32c}$$

The eigenvalues for $\frac{1}{2}M$ and Y may be read from Fig. 2.

(2) The direct products of the two triplets is a nonet

$$[3] \otimes [3] : \{f_1 f_1, f_1 f_2, f_1 f_3, f_3 f_3, f_2 f_3, f_2 f_2, f_1 f_2, f_1 f_3, f_2 f_3\}. \tag{33}$$

One has

$$\mathscr{L}\{[3] \otimes [3]\} = 2\{[3] \otimes [3]\}, \tag{34a}$$

$$\mathscr{K}\{[3] \otimes [3]\} = \tfrac{40}{3}\{[3] \otimes [3]\}. \tag{34b}$$

Their m and y eigenvalues may be read from Fig. 3. The reduction technique above may be extended to obtain the higher-order modes.

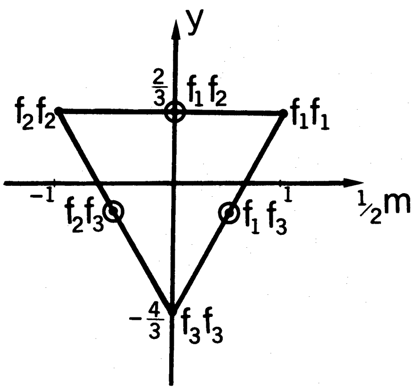

Fig. 3. $1/2m$–y diagram of $[3] \otimes [3]$ multiplet. See Eq. (33), and comments thereof

5.2. Normal modes through $SL(3, c)$

The largest noncompact semisimple subgroup of symmetries of $\mathscr{L}$ is $SL(3, c)$. It is of rank 4. That is, it has four mutually commuting operators and four Casimir operators, eight together. However, only six of this set of eight operators can be independent. We proposes the set $\{\mathscr{L}, \Lambda, \mathscr{M}_{\pm}, \mathscr{Y}_{\pm}\}$, where

$$\Lambda = \mathrm{i}(\boldsymbol{J} \cdot \boldsymbol{I} + \boldsymbol{L} \cdot \boldsymbol{K} + \sqrt{3} MR + \sqrt{3}\, YS), \tag{35a}$$

$$\mathscr{M}_{\pm} = \tfrac{1}{4}(M \pm \mathrm{i}R), \tag{35b}$$

$$\mathscr{Y}_{\pm} = \tfrac{1}{4}(Y \pm \mathrm{i}S). \tag{35c}$$

The common eigenfunctions of this set are

$$f^{\lambda\mu\nu}_{\alpha\beta\gamma} = z_1^{\alpha} z_2^{\beta} z_3^{\gamma} z_1^{*\lambda} z_2^{*\mu} z_3^{*\nu}, \tag{36a}$$

$$z_j = p_j + \mathrm{i}q_j, \quad j = 1, 2, 3, \tag{36b}$$

with the following eigenvalues

$$\mathscr{L} f^{\lambda\mu\nu}_{\alpha\beta\gamma} = [\alpha + \beta + \gamma - (\lambda + \mu + \nu)] f^{\lambda\mu\nu}_{\alpha\beta\gamma}, \tag{37a}$$

$$\Lambda f^{\lambda\mu\nu}_{\alpha\beta\gamma} = 4\{[\alpha + \beta + \gamma - (\lambda + \mu + \nu)] + \frac{1}{3}[(\alpha - \beta + \gamma)^2 - (\lambda + \mu + \nu)^2]\} f^{\lambda\mu\nu}_{\alpha\beta\gamma}, \tag{37b}$$

$$\mathscr{M}_{+} f^{\lambda\mu\nu}_{\alpha\beta\gamma} = \tfrac{1}{2}(\alpha - \beta) f^{\lambda\mu\nu}_{\alpha\beta\gamma}, \tag{37c}$$

$$\mathscr{M}_{-} f^{\lambda\mu\nu}_{\alpha\beta\gamma} = \tfrac{1}{2}(\nu - \mu) f^{\lambda\mu\nu}_{\alpha\beta\gamma}, \tag{37d}$$

$$\mathscr{Y}_{+} f^{\lambda\mu\nu}_{\alpha\beta\gamma} = \tfrac{1}{3}(\alpha + \beta - 2\gamma) f^{\lambda\mu\nu}_{\alpha\beta\gamma}, \tag{37e}$$

$$\mathscr{Y}_{-} f^{\lambda\mu\nu}_{\alpha\beta\gamma} = \tfrac{1}{3}(2\lambda - \mu - \nu) f^{\lambda\mu\nu}_{\alpha\beta\gamma}. \tag{37f}$$

It was mentioned in Paper II that the Hilbert space, $\mathscr{H}$, has two analytic and coanalytic subspaces, spanned by the analytic and coanalytic functions of z_i and z_i^*, respectively. One may note that the eigenvalues of $\mathscr{M}_{-}$ and $\mathscr{Y}_{-}$ are zero in the analytic subspace and those of $\mathscr{M}_{+}$ and $\mathscr{Y}_{+}$ are zero in the coanalytic one. The two subspace, are invariant under the symmetry group $SL(3, c)$. Of course, the full space is larger than the analytic and coanalytic subspaces and contains the nonanalytic functions of $z_i z_i^*$ as well.

Appendix: Constants of motion and symmetry transformations

As for Killing's vectors in general relativity where $\xi_i \dot{q}_i = \xi_i p_i$ are constants along geodesics, here one can show, using Eqs.

(5a)–(5c), that $\xi_i \dot{q}_i + \eta_i \dot{p}_i = \xi_i p_i - \eta_i q_i$ are constants along the phase space orbits. Using Eqs. (8a) and (8b), one gets eighteen constants of motion. Nine of them, corresponding to symmetric a_{ij}'s and antisymmetric b_{ij}' s, are zero. The rest of them belonging to antisymmetric a_{ij}'s and symmetric b_{ij}'s are components of angular momentum, $l_i = \varepsilon_{ijk} q_j p_k$, and the symmetric tensor $A_{ij} = q_i q_j + p_i p_j$. These constants, of motion correspond to the nine canonical transformations of Sect. 4.2.2 which leave the Hamiltonian invariant (Goldstein 1980).

References

Barut A.O., 1965, Phys. Rev. B 139, 1433

Fradkin D.M., 1965, Am. J. Phys. 33, 207

Goldstein H., 1980, Classical Mechanics (2nd edn.). Addison-Wesley Reading, MA

Greiner W., Muller B., 1989, Quantum Mechanics Symmetries (Vol. 2). Springer, Berlin

Haskell T.G., Wybourne B.G., 1973, Proc. Roy. Soc. (London), A 334, 541

Hwa R.C., Nuyts J., 1966, Phys. Rev. 145, 1188

Jauch J.M., Hill E.L., 1940, Phys. Rev. 57, 641

Prigogine I., 1962, Non-equilibrium Statistical Mechanics. Wiley, New York

Sobouti Y., 1989a, A&A 210, 18 (Paper I)

Sobouti Y., 1989b, A&A 214, 83 (Paper II)

Sobouti Y., Samimi J., 1989, A&A 214, 92 (Paper III)

Wybourne B.G., 1974, Classical Groups for Physicists, Wiley, New York