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Liouville's equation

V. The full symmetries of r^{-1}-potentials*

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Abstract. A systematic study of the symmetries of Liouville's equation for r^{-1}-potential is presented. The canonical transformations in phase space which leave the hamiltonian invariant turn out to be the full symmetry transformations of Liouville's operator, as well. The symmetry group is SO(4). A maximal set of mutually commuting operators, and subsequently, a classification of the eigensolutions of Liouville operator is proposed. The Kustaanheimo–Stiefel transformation is used to show that this SO(4) is isomorphic to a constrained SU(4) and contains all symmetries of Liouville's equation for r^{-1}-potential.

Key words: galaxies – dynamics and evolution – Liouville's equation – symmetries and normal modes

1. Introduction

The symmetry group of Schrodinger equation for hydrogen atom has been considered by many investigators. It is known that the minimal group which has representations completely spanning the states of individual degenerate levels of hydrogen atom is SO(4) (Fock 1935). Also a larger group which removes the degeneracies of the energy levels, and contains a set of operators determining the transition probabilities between the states has been considered by Bander & Itzykson (1966), Barut & Kleinert (1967), Englefield (1972), Wybourne (1974), and others.

In Paper IV of this series (Sobouti & Dehghani 1992) a systematic method was developed to explore the symmetries of Liouville's equation for an arbitrary potential and was applied to explore the details for quadratic potentials. In this paper we do the same for r^{-1}-potential and find that the symmetry group is SO(4). Kustaanheimo–Stiefel (KS) transformation (Kustaanheimo & Stiefel 1965) maps the three dimensional Kepler problem into a restricted four dimensional harmonic oscillator problem. The map is of course nonbijective. We use KS transformation to show that the SO(4) is indeed the largest symmetry group of Liouville's equation for r^{-1}-potential. The symmetry operators in KS transformation have simpler form and are ecstatically appealing.

In Sect. 2 we outline some pertinent points from Paper IV. In Sect. 3 we find six canonical transformations in phase space which leave Liouville's equation invariant. In Sect. 4 we elaborate on the invariance of hamiltonian under these transformations, and associate each transformation with one constant of motion. The latter are of course the six components of angular momentum and of Runge–Lenz vector of Kepler's problem. In Sect. 5 we derive the SO(4) symmetry of Liouville's equation and use KS transformation to show that SO(4) is indeed the largest symmetry group. The details of KS transformation, the isomorphism of SO(4) and the constrained SU(4), and the associated generators are presented in the appendix. Section 6 contains closing comments.

2. Background review

Let Liouville's equation be written as

$$\mathscr{L}f = \mathrm{i}\frac{\partial f}{\partial t}, \quad \mathscr{L} = -\mathrm{i}\left(p_i\frac{\partial}{\partial q_i} - \frac{\partial U}{\partial q_i}\frac{\partial}{\partial p_i}\right), \tag{1}$$

where $(\boldsymbol{q}, \boldsymbol{p})$ is the collection of configuration and momentum coordinates respectively, $U(\boldsymbol{q})$ is the potential, $\mathscr{L}$ is Liouville's operator and $f(\boldsymbol{q}, \boldsymbol{p}, t)$ is in general a complex-valued function of $(\boldsymbol{q}, \boldsymbol{p}, t)$. The latter is a member of function space, a Hilbert space $\mathscr{H}$, in which the inner product is defined as

$$(f, g) = \int f^* g \,\mathrm{d}\boldsymbol{q}\,\mathrm{d}\boldsymbol{p} < \infty, \quad f, g \in \mathscr{H}. \tag{2}$$

The reason for including i in Eq. (1) is to render $\mathscr{L}$ hermitian. That is, $(\mathscr{L}f, g) = (f, \mathscr{L}g)$; $f, g \in \mathscr{H}$. See Prigogine (1962) and Sobouti (1989a) for more details. As in Paper IV we look for infinitesimal transformations of the

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type

$$q_i' = q_i + \varepsilon \xi_i(\boldsymbol{q}, \boldsymbol{p}), \tag{3a}$$

$$p_i' = p_i + \varepsilon \eta_i(\boldsymbol{q}, \boldsymbol{p}), \tag{3b}$$

which leave Liouville's operator form-invariant, where ε is an infinitesimal parameter. The form-invariance of Liouville's operator leads to the following differential equation for ξ_i and η_i.

$$\mathscr{L}\xi_i + \mathrm{i}\eta_i = 0, \tag{4a}$$

$$\mathscr{L}\eta_i - \mathrm{i}\xi_j \frac{\partial^2 U}{\partial q_j \partial q_i} = 0. \tag{4b}$$

Or, on eliminating η_i between the two equations,

$$\mathscr{L}^2 \xi_i - \xi_j \frac{\partial^2 U}{\partial q_j \partial q_i} = 0.$$

As a consequence of these transformations, the infinitesimal change in a phase space function is $\mathrm{i}\varepsilon\chi f(\boldsymbol{q}, \boldsymbol{p}, t)$; $f \in \mathscr{H}$, where the generator χ is given by

$$\chi = -\mathrm{i}\left(\xi_i \frac{\partial}{\partial q_i} + \eta_i \frac{\partial}{\partial p_i}\right). \tag{5a}$$

It satisfies the following commutation rule:

$$[\mathscr{L}, \chi] = 0. \tag{5b}$$

3. Symmetry transformation for r^{-1}-potential

For $U = -r^{-1} = -(q_i \cdot q_i)^{-1/2}$, Eq. (4c) becomes

$$\mathscr{L}^2 \xi_i = \frac{\xi_i}{r^3} - \frac{3(\boldsymbol{q} \cdot \boldsymbol{\xi}) q_i}{r^5}, \tag{6a}$$

where

$$\mathscr{L} = -\mathrm{i}\left(p_i \frac{\partial}{\partial q_i} - \frac{q_i}{r^3} \frac{\partial}{\partial p_i}\right). \tag{6b}$$

Equations (6a) and (6b) have solutions linear and quadratic in q_i and p_i. These are discussed separately in the following sections.

3.1. Solutions linear in q_i and p_i

From Papers I and IV we know the following solutions:
(a)

$$\xi_i = p_i, \tag{7a}$$

$$\eta_i = -\frac{q_i}{r^3}. \tag{7b}$$

Interpreting ε of Eqs. (3a) and (3b) as an infinitesimal time interval and noting that p_i and $-q_i/r^3$ are the velocity and acceleration, respectively (mass = 1), the transformation induced by Eqs. (7a) and (7b) turns out to be an infinitesimal time evolution along a phase space trajectory. The generator χ, corresponding to Eqs. (7a) and (7b) is the Liouville's operator itself.
(b)

$$\xi_j = \varepsilon_{ijk} q_k, \tag{8a}$$

$$\eta_j = \varepsilon_{ijk} p_k, \tag{8b}$$

where ε_{ijk} is the Levi Civita symbol. There are three independent sets of Eqs. (8a) and (8b) for the free index $i = 1, 2, 3$. Equations (8a) and (8b) express infinitesimal rotation of the configuration and momentum coordinates about the ith axis by an angle ε. The corresponding generators, χ of Eq. (5a), are

$$J_i = J_i^\dagger = -\varepsilon_{ijk}\left(p_j \frac{\partial}{\partial p_k} + q_j \frac{\partial}{\partial q_k}\right). \tag{9a}$$

They have the SO(3) algebra

$$[J_i, J_j] = \mathrm{i}\varepsilon_{ijk} J_k. \tag{9b}$$

The corresponding Casimir operator is J^2, with $[J^2, J_i] = 0$. These three transformations form a subgroup of the full symmetry group. They hold for any spherically symmetric potential (Papers I and II) and also for the linearized Liouville equation pertaining to self gravitating system (Paper III).

3.2. Solutions quadratic in q_i and p_i

Now we consider solutions of the type

$$\xi_i = a_{ijk} q_j p_k + b_{ijk} q_j q_k + c_{ijk} p_j p_k. \tag{10}$$

One finds that there exist only three independent solutions of this kind as follows

$$\xi_i^{(a)} = (\boldsymbol{q} \cdot \boldsymbol{p})\, \delta_i^a + p_a q_i - 2 q_a p_i, \quad a = 1, 2, 3. \tag{11a}$$

The corresponding $\eta_i^{(a)}$'s are

$$\eta_i^{(a)} = \left(p^2 - \frac{1}{r}\right) \delta_i^a - p_a p_i + \frac{q_a q_i}{r^3}, \quad a = 1, 2, 3. \tag{11b}$$

The transformations belonging to these three different solutions are

$$q_i' = q_i \varepsilon_a [(\boldsymbol{q} \cdot \boldsymbol{p}) \delta_i^a + p_a q_i - 2 q_a p_i], \tag{12a}$$

$$p_i' = p_i + \varepsilon_a\left[\left(p^2 - \frac{1}{r}\right) \delta_i^a - p_a p_i + \frac{q_a q_i}{r^3}\right] \quad \text{(no sum on } a\text{)}. \tag{12b}$$

There are three sets of Eqs. (12a) and (12b) for $a = 1, 2, 3$. One may easily verify that these transformations are canonical, that is

$$[q_i', q_j']_{\mathrm{P}} = [p_i', p_j']_{\mathrm{P}} = 0, \tag{13a}$$

$$[q_i', p_j']_{\mathrm{P}} = \delta_{ij}, \tag{13b}$$

where $[\ldots, \ldots]_{\mathrm{P}}$ is the Poisson bracket. The corresponding generators, χ, are

$$A_i = A_i^\dagger = -\mathrm{i}\left\{(\boldsymbol{q}\cdot\boldsymbol{p})\frac{\partial}{\partial q_i} + p_i q_j \frac{\partial}{\partial q_j} - 2q_i p_j \frac{\partial}{\partial q_j} + \left(p^2 - \frac{1}{r}\right)\frac{\partial}{\partial p_i} - p_i p_j \frac{\partial}{\partial p_j} + \frac{q_i}{r^3} q_j \frac{\partial}{\partial p_j}\right\}. \quad (14)$$

4. Symmetry transformations of hamiltonian

In addition to Liouville's operator, the hamiltonian is also invariant under the canonical transformations of Sect. 3. It is well known that canonical transformations which leave hamiltonian unchanged lead to constants of motion (Goldstein 1980). The constant corresponding to Eqs. (7a) and (7b) is the hamiltonian itself. Those for Eqs. (8a) and (8b) are the three component of angular momentum. Finally the constants for the quadratic transformations of Eqs. (11a) and (11b) are the three components of the Runge–Lenz vector. They are as follows:

$$a_i = \varepsilon_{ijk} p_j l_k - \frac{q_i}{r} = \left(p^2 - \frac{1}{r}\right) q_i - (\boldsymbol{q}\cdot\boldsymbol{p}) p_i, \quad (15)$$

where l_k is the kth component of angular momentum. To the best of our knowledge the canonical transformations which lead to the constancy of Runge–Lenz vector has not been pointed out before.

5. Structure of the symmetry group

The six operators J_i and A_i commute with Liouville's operator. It is a matter of straightforward calculation to show that

$$[A_i, A_j] = -2\mathrm{i}\varepsilon_{ijk}(HJ_k + l_k \mathscr{L}), \quad (16a)$$

and

$$[J_i, A_j] = \mathrm{i}\varepsilon_{ijk} A_k. \quad (16b)$$

If one defines M_i as

$$M_i = \frac{1}{\sqrt{-2H}}\left(A_i + \frac{a_i}{2H}\mathscr{L}\right), \quad (17)$$

where a_i is the ith component of Runge–Lenz vector, then it is easy to verify

$$[M_i, M_j] = \mathrm{i}\varepsilon_{ijk} J_k, \quad (18a)$$

$$[J_i, M_j] = \mathrm{i}\varepsilon_{ijk} M_k. \quad (18b)$$

Thus, the symmetry group of Liouville's equation for r^{-1}-potential has SO(4) algebra as expressed in Eqs. (9b) and (18). The corresponding Casimir operators are $J^2 + M^2$ and $\boldsymbol{J}\cdot\boldsymbol{M}$.

In Sect. 3 we explored linear and quadratic solutions of Eq. (6a) and arrived at the SO(4) symmetry of Liouville's operator. One may ask whether there are other solutions to Eq. (6a) and, therefore, a larger group of symmetries. To show that the solutions of Sect. 3 are complete and SO(4) is indeed the largest group, we have investigated the problem once more by means of Kustaanheimo–Stiefel transformation. This transformation maps a six dimensional phase space $(\boldsymbol{q}, \boldsymbol{p})$ to an eight dimensional phase space $(\boldsymbol{u}, \boldsymbol{v})$, simplifies the form of Liouville's operator, and allows the procedure of Paper IV for the three dimensional oscillators to be used profitably. We have been able to find fifteen operators commuting with Liouville's operator in $(\boldsymbol{u}, \boldsymbol{v})$ space. They obey the SU(4) algebra. But KS transformation is not one to one. This imposes a constraint on both solutions of four dimensional oscillator and its symmetry transformations. This aspect is worked out explicitly in the appendix and it is shown that the constrained SU(4) is isomorphic to SO(4).

6. Closing comments

This section reflects a contemplation on a future follow up of the line of thought pursued in this series of papers. What are the eigensolutions of Liouville's equation good for and what possible astronomical purpose they might serve?

Let us concentrate on the zero eigenvalue of $\mathscr{L}$. This is highly degenerate with any arbitrary function of the constants of motion (energy, angular momentum, Runge–Lenz vector) as a time independent eigenfunction and, thus, an equilibrium state of the system. To unravel the degeneracy, that is, to impose some sort of classification on the possible equilibrium configurations one may introduce, along with, $\mathscr{L}$, a set of mutually commuting operators, and seek their simultaneous eigensolutions. One alternative is the following.

For a group of rank ℓ, one can introduce 2ℓ mutually commuting operators. Half of them are a set of operators from the generators of the group and the other half are the ℓ Casimir operators. The eigenvalues of Casimir operators classify the eigensolutions. Indeed the eigensolutions with the same eigenvalues for Casimir operators form an invariant subspace. The continuous group, SO(4), is of rank 2. Thus, one can introduce four operators which commute with each other and with $\mathscr{L}$. If one defines $K_i = \frac{1}{2}(J_i + M_i)$ and $K_i' = \frac{1}{2}(J_i - M_i)$, then

$$[K_i, K_j] = \mathrm{i}\varepsilon_{ijk} K_k, \quad [K_i', K_j'] = \mathrm{i}\varepsilon_{ijk} K_k', \quad (19a)$$

$$[K_i, K_j'] = 0. \quad (19b)$$

That is, K_i and K_i' both have an angular momentum algebra. Thus, one can choose $\{K^2, K'^2, K_3, K_3'\}$ as the commuting set. The simultaneous eigenfunctions of this set are

$$K^2 f_{km}^{k'm'} = k(k+1) f_{km}^{k'm'}, \qquad K_3 f_{km}^{k'm'} = m f_{km}^{k'm'}, \quad (20a)$$

$$K'^2 f_{km}^{k'm'} = k'(k'+1) f_{km}^{k'm'}, \qquad K_3' f_{km}^{k'm'} = m' f_{km}^{k'm'}. \quad (20b)$$

The dimension of the invariant subspace labeled by k and k' is evidently $(2k+1)(2k'+1)$. The whole set of functions belonging to an invariant subspace may be obtained from any one by stepping through the set with raising and lowering operators $K_\pm = K_1 \pm iK_2$ and $K'_\pm = K'_1 \pm iK'_2$.

In summary, any function of the constants of motion that satisfies Eqs. (20) is a possible equilibrium state of the system, characterized by the discrete parameters (k, m, k', m'). For example, based on symmetry considerations, any spherically symmetric and isotropic distribution will belong to (0, 0, 0, 0) class. Spheroidal and ellipsoidal configurations with non-isotropic distribution of momenta will have a different set of parameters. We hope to present this type of analysis in the near future.

Non-zero spectrum of $\mathscr{L}$ is continuous. See Prosser (1969) and Spohn (1975). The corresponding eigenfunctions represent time dependent states, which according to Prosser eventually relax to some static state. Any such solution may still be designated by a set (k, m, k', m') and keep this designation unchanged in the course of time evolution.

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Appendix: constrained SU(4) as the symmetry group of Liouville's equation

The Kustaanheimo–Stiefel (KS) transformation connects a three dimensional Kepler problem with that of a restricted four dimensional harmonic oscillator (Kibler & Winternitz 1988; Kibler et al 1986; Chen 1985; Cerdeira 1985; Boiteux 1973, 1974, 1982); the transformation is

$$q_i = A_{i\alpha} u_\alpha, \quad i = 1, 2, 3, \quad \alpha = 1, 2, 3, 4, \tag{A1a}$$

where $A_{i\alpha}$ is the following 4×4 matrix

$$A(u) = \begin{pmatrix} u_3 & -u_4 & u_1 & -u_2 \\ u_4 & u_3 & u_2 & u_1 \\ u_1 & u_2 & -u_3 & -u_4 \\ u_2 & -u_1 & -u_4 & u_3 \end{pmatrix}. \tag{A1b}$$

The corresponding momentum transformation is

$$p_i = \frac{1}{u^2} A_{i\alpha} v_\alpha, \tag{A1c}$$

where v_α denotes the canonical momentum conjugate to u_α. Liouville's operator in this eight dimensional space becomes

$$\mathscr{L} = -\frac{i}{u^2}\left(v_\alpha \frac{\partial}{\partial u_\alpha} + 8Hu_\alpha \frac{\partial}{\partial v_\alpha}\right), \quad H = \frac{v^2 - 8}{8u^2}. \tag{A2}$$

The transformation from $(\boldsymbol{q}, \boldsymbol{p})$ to $(\boldsymbol{u}, \boldsymbol{v})$ is not one to one. All points in (u_1, u_2), (u_4, u_3), (v_1, v_2) and (v_4, v_3)-planes, obtained one from another by rotations through an arbitrary angle ϕ, represent one and the same point in $(\boldsymbol{q}, \boldsymbol{p})$ space. A function $f(\boldsymbol{u}, \boldsymbol{v}, t)$ to be single-valued in $(\boldsymbol{q}, \boldsymbol{p})$ space should remain invariant under such rotations. The corresponding generator and the constraint is as follows:

$$-i\left(u_1 \frac{\partial}{\partial u_2} - u_2 \frac{\partial}{\partial u_1} + u_4 \frac{\partial}{\partial u_3} - u_3 \frac{\partial}{\partial u_4} + v_1 \frac{\partial}{\partial v_2} - v_2 \frac{\partial}{\partial v_1} + v_4 \frac{\partial}{\partial v_3} - v_3 \frac{\partial}{\partial v_4}\right) f = Rf = 0. \tag{A3}$$

Thus, $f(\boldsymbol{u}, \boldsymbol{v}, t)$, in addition to being a solution of Eqs. (1) and (A2), should also satisfy Eq. (A3).

Following the procedure of Paper IV, we have been able to introduce fifteen operators, χ_μ, $\mu = 1, 2, \ldots, 15$, commuting with Liouville's operator of Eq. (A2). They obey SU(4) algebra. From $[\mathscr{L}, \chi_\mu] = 0$, we note that if f is a solution of Liouville's equation then so is $\chi_\mu f$. The constraint conditions for these solutions are

$$Rf = 0, \tag{A4a}$$

$$R(\chi_\mu f) = 0. \tag{A4b}$$

Operating on Eq. (A4a) by χ_μ and subtracting it from Eq. (A4b), we obtain

$$(R\chi_\mu - \chi_\mu R) f = [R, \chi_\mu] f = 0. \tag{A5}$$

That is, the operators of symmetry group should commute with both $\mathscr{L}$ and R. Thus, one must look for the largest subalgebra of SU(4) which commutes with R. We have found that of the fifteen χ_μ only six commute with R, and have a closed algebra. They are no other than the J_i and M_i of Eqs. (9a) and (17). Once more, we arrive at the conclusion that the largest symmetry group is SO(4), a subgroup of SU(4). Expressed in $(\boldsymbol{u}, \boldsymbol{v})$ coordinates J_i and M_i are

$$J_i = \frac{i}{2}\left\{\varepsilon_{ijk}\left(v_j \frac{\partial}{\partial v_k} + u_j \frac{\partial}{\partial u_k}\right) + v_i \frac{\partial}{\partial v_4} - v_4 \frac{\partial}{\partial v_i} + u_i \frac{\partial}{\partial u_4} - u_4 \frac{\partial}{\partial u_i}\right\},$$

$$M_1 = \frac{-i}{2\sqrt{-2H}}\left\{v_1 \frac{\partial}{\partial u_3} + v_3 \frac{\partial}{\partial u_1} - v_2 \frac{\partial}{\partial u_4} - v_4 \frac{\partial}{\partial u_2} + 8H\left(u_1 \frac{\partial}{\partial v_3} + u_3 \frac{\partial}{\partial v_1} - u_2 \frac{\partial}{\partial u_4} - u_4 \frac{\partial}{\partial v_2}\right) - 4i\left(u_1 u_3 - u_2 u_4 + \frac{v_1 v_3 - v_2 v_4}{8H}\right)\mathscr{L}\right\},$$

$$M_2 = \frac{-i}{2\sqrt{-2H}}\left\{v_2 \frac{\partial}{\partial u_3} + v_3 \frac{\partial}{\partial u_2} + v_1 \frac{\partial}{\partial u_4} + v_4 \frac{\partial}{\partial u_1} + 8H\left(u_2 \frac{\partial}{\partial v_3} + u_3 \frac{\partial}{\partial v_2} + u_1 \frac{\partial}{\partial v_4} + u_4 \frac{\partial}{\partial v_1}\right) - 4i\left(u_2 u_3 + u_1 u_4 + \frac{v_2 v_3 + v_1 v_4}{8H}\right)\mathscr{L}\right\},$$

$$M_3 = \frac{-i}{2\sqrt{-2H}}\left\{v_1 \frac{\partial}{\partial u_1} + v_2 \frac{\partial}{\partial u_2} - v_3 \frac{\partial}{\partial u_3} - v_4 \frac{\partial}{\partial u_4}\right.$$

$$+8H\left(u_1\frac{\partial}{\partial v_1}+u_2\frac{\partial}{\partial v_2}-u_3\frac{\partial}{\partial v_3}-u_4\frac{\partial}{\partial v_4}\right)$$
$$-2i\left(u_1^2+u_2^2-u_3^2-u_4^2+\frac{v_1^2+v_2^2-v_3^2-v_4^2}{8H}\right)\mathscr{L}\Bigg\}.$$

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