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Dynamical group of Liouville's equation for quadratic potentials*

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Abstract. A dynamical symmetry group of Liouville's equation for quadratic potentials is obtained. A complete set of mutually commuting operators and the ladder operators to generate the simultaneous eigenfunctions of the set are given.

Key words: stellar dynnamics – methods: analytical

1. Introduction

More often than not astronomers maintain that the equilibrium distribution functions of stellar systems in dynamical time scales are governed by Liouville's equation. This equation, however, possesses eigensolutions giving rise to time dependent changes of definite patterns in density, bulk motion and other physical parameters that may be amenable to observations. Nevertheless, these eigensolutions are highly degenerate. For, any eigensolution of Liouville's equation multiplied by an arbitrary function of constants of motion is again an eigensolution belonging to the same eigenvalue. A systematic method of disentangling these degeneracies can be achieved through the use of the symmetry group. In a series of papers, Sobouti (1989a, b), Sobouti & Samimi (1989), Sobouti & Dehghani (1992) and Dehghani & Sobouti (1993), hereafter referred to as Papers I–V, study the symmetries of the Liouville and the linearized Liouville-Poisson equations. In Papers I–III it was shown that for spherically symmetric equilibrium potentials the symmetry group is O(3). In Papers IV and V a systematic method was introduced to find the symmetries of Liouville's equation for an arbitrary potential. The method was then applied to quadratic and r^{-1} potentials.

In this paper we seek the *dynamical* symmetries of Liouville's equation. In Sects. 2 and 3 we discuss the terms *invariance* and *dynamical* groups, write down an algorithm for finding the dynamical group for quadratic potentials and analyze the case of spherical symmetry in full details. In Sects. 4 and 5 we introduce a complete set of six mutually commuting operators and give the appropriate ladders to generate the whole set of the simultaneous eigensolutions of the commuting set.

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2. Invariance and dynamical groups of Liouville's equation

Liouville's equation may be written as

$$\mathscr{D} f \equiv \left(i\frac{\partial}{\partial t} - \mathscr{L}\right) f = 0;\ \mathscr{L} = -i\left(p_i\frac{\partial}{\partial q_i} - \frac{\partial U}{\partial q_i}\frac{\partial}{\partial p_i}\right), \quad (1)$$

where $(\mathbf{q}, \mathbf{p})$ is the collection of configuration and momentum coordinates respectively, $U(\mathbf{q})$ is the potential, $\mathscr{L}$ is Liouville's operator and $f(\mathbf{q}, \mathbf{p}, t)$ is in general a complex-valued function of $(\mathbf{q}, \mathbf{p}, t)$. The latter is a member of a function space, a Hilbert space $\mathscr{H}$, in which the inner product is defined as

$$(f, g) = \int f^* g \,\mathrm{d}\mathbf{q}\mathrm{d}\mathbf{p} < \infty; \qquad f, g \in \mathscr{H}. \quad (2)$$

The reason for including i in Eq. (1) is to render $\mathscr{L}$ hermitian. That is $(\mathscr{L} f, g) = (f, \mathscr{L} g)$; $f, g \in \mathscr{H}$. See Prigogine (1962), and Papers I, IV and V for more details. Let $\{\chi_\alpha\}$ be the generators of the *invariance* group of Liouville's operator, that is the set of all operators commuting with $\mathscr{L}$

$$[\mathscr{L}, \chi_\alpha] = 0. \quad (3a)$$

We recall from Paper IV that $\{\chi_\alpha\}$ have a closed Lie algebra. i.e.

$$[\chi_\alpha, \chi_\beta] = C^\gamma_{\alpha\beta}\chi_\gamma, \quad (3b)$$

where $C^\gamma_{\alpha\beta}$ are the structure constants. This algebra contains no operator which can connect the eigenfunctions of Liouville's operator with different eigenvalues. In other words each irreducible representation of the algebra is realized on the solutions belonging to one and only one eigenvalue.

In analogy, the invariance group of $\mathscr{D}$ of Eq. (1), hereafter referred to as the *dynamical* group of Liouville's equation, consists of the set of operators $\{\mathscr{R}_\mu\}$ which commute with $\mathscr{D}$ and have a closed algebra

$$[\mathscr{D}, \mathscr{R}_\mu] = 0, \tag{4a}$$

$$[\mathscr{R}_\mu, \mathscr{R}_\nu] = C^\delta_{\mu\nu}\mathscr{R}_\delta. \tag{4b}$$

Each irreducible representation of this dynamical group is realized in the space of all the solutions of Eq. (1). In other words, $\mathscr{R}_\mu$ contains the ladder operators connecting the eigensolutions belonging to different eigenvalues.

3. Dynamical symmetries for quadratic potentials

In the quantum mechanical context, the invariance group of the hamiltonian for three dimensional harmonic potential is $SU(3)$ (Jauch & Hill 1940; Fradkin 1965). This is the minimal group that completely spans the states of a given energy of a harmonic oscillator. It, however, does not allow transition between different energy levels. A larger dynamical group which is free of this limitation has been investigated by Barut (1964, 1965), Hwa & Nuyts (1966), Malkin et al. (1971), and Haskell & Wybourne (1973). Here we are concerned with the dynamical group of Liouville's equation in a six dimensional phase space, and use a generalized version of the method of Malkin et al. for Schrodinger's equation.

Let us consider the quadratic potential, $U = \frac{1}{2}k_{ij}(t)q_iq_j$. The time dependence of k_{ij} is, at this stage, of academic interest only. And it will be retained as long as it does not give rise to serious mathematical complications. Liouville's operator of Eq. (1) may be written as

$$\mathscr{L} = \mathbf{Q}^\dagger\mathbf{M}\mathbf{Q}, \tag{5a}$$

where $\mathbf{Q}$ is the following column vector

$$(Q_1, Q_2, ..., Q_{12}) \equiv \left(q_j, p_j, -i\frac{\partial}{\partial q_j} - i\frac{\partial}{\partial p_j}\right); \quad j = 1, 2, 3, \tag{5b}$$

and $\mathbf{M}$ is the real 12×12 matrix, below

$$\mathbf{M} = \frac{1}{2}\begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & -\mathbf{k} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} \\ -\mathbf{k} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix}. \tag{5c}$$

Here, $\mathbf{0}, \mathbf{1}$ and $\mathbf{k}$ are the 3×3 zero, unit and k_{ij} matrices. Let us examine the linear hermitian operators,

$$\mathscr{R}_\mu = \Lambda_{\mu\nu}(t)Q_\nu; \qquad \mu, \nu = 1, 2, ..., 12 \tag{6}$$

as possible members of the dynamical group. We shall determine $\Lambda_{\mu\nu}$ by, requiring Eq. (4a) to be satisfied. Later we shall see that because of the full integrability of Liouville's equation in this case, there are twelve such operators. To determine $\mathbf{\Lambda}$ we proceed as follows. The components of $\mathbf{Q}$ satisfy the commutation rules

$$[Q_\mu, Q_\nu] = -i\Delta_{\mu\nu}, \tag{7a}$$

where

$$\Delta = \begin{pmatrix} \mathbf{0} & \mathbf{0} & -\mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & -\mathbf{1} \\ \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} \end{pmatrix}. \tag{7b}$$

Using these commutators and the invariance requirement of Eq. (4a), we obtain the following evolution equation for $\mathbf{\Lambda}$

$$\frac{\partial\Lambda}{\partial t} = \Lambda(t)\mathbf{N}(t), \tag{8}$$

where $\mathbf{N} = \Delta(\mathbf{M} + \mathbf{M}^\dagger)$.

By construction the operators $\mathscr{R}_\mu$ of Eq. (6) are members of the dynamical group. They do not, however exhaust all possibilities. Any linear combination of the pair products $\mathscr{R}_\mu\mathscr{R}_\nu$ belongs to this algebra. In particular there is a subalgebra of time independent combinations which constitute the invariance group of $\mathscr{L}$, as discussed in Paper IV. There is no need to consider products of three or more $\mathscr{R}'_\mu s$. For, they do not have a closed Lie algebra and do not provide new information.

For the case of spherically symmetric potential with time independent coefficients, $U = \frac{1}{2}q_iq_i$, one obtains

$$\mathbf{N} = \begin{pmatrix} \mathbf{0} & -\mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & -\mathbf{1} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} \end{pmatrix}. \tag{9}$$

Equation (8) then gives

$$\mathbf{\Lambda} = e^{\mathbf{N}t} = \mathbf{1}\cos t + \mathbf{N}\sin t, \tag{10}$$

where we have used the relations $\mathbf{N}^{2n} = (-1)^n\mathbf{1}$ and $\mathbf{N}^{2n+1} = (-1)^n\mathbf{N}$ and have expanded the exponential term. From this, the twelve hermitian operators $\mathscr{R}_\mu$ follow

$$\mathscr{R}_i = q_i\cos t - p_i\sin t, \tag{11a}$$

$$\mathscr{R}_{i+3} = q_i\sin t + p_i\cos t, \tag{11b}$$

$$\mathscr{R}_{i+6} = -i\cos t\frac{\partial}{\partial q_i} + i\sin t\frac{\partial}{\partial p_i}, \tag{11c}$$

$$\mathscr{R}_{i+9} = -i\sin t\frac{\partial}{\partial q_i} + i\cos t\frac{\partial}{\partial p_i}. \tag{11d}$$

Instead of $\mathscr{R}$'s, however, we find it convenient to use the following non hermitian combinations along with their complex conjugates and adjoints

$$T_i = \frac{1}{\sqrt{2}}[\mathscr{R}_{i+3} + \mathscr{R}_{i+6} - i(\mathscr{R}_i + \mathscr{R}_{i+9})] = \frac{1}{\sqrt{2}}\left(p_i - iq_i + \frac{\partial}{\partial p_i} - i\frac{\partial}{\partial q_i}\right)e^{it}. \tag{12}$$

By virtue of Eqs. (7) or by direct manipulations, one obtains

$$[T_i, T_j^\dagger] = \delta_{ij}, \qquad [T_i^*, T^{*\dagger}_j] = \delta_{ij}, \tag{13a}$$

$$[T_i, T^{*\dagger}_j] = [T_i^*, T_j^\dagger] = [T_i, Tj] = [T_i^*, T_j^*] = 0. \tag{13b}$$

The dynamical group now contains the twelve operators $T_i, T_i^\dagger, T^*{}_i, T^*{}_i^\dagger$, and their various pair products. From the pair products we will construct sets of hermitian and mutually commuting operators. The non hermitian T's themselves will serve as ladders to go from one eigensolution to others.

4. A complete set of mutually commuting operators

A typical eigenfunction of $\mathscr{L}$, of the form $\prod_{i,j=1}^{3}(p_i + iq_i)^{n_i}(p_j - iq_j)^{n'_j}$, is characterized by six eigenvalues n_i and $n'_j; i,j = 1,2,3$, and belongs to the highly degenerate eigenvalue $\sum_{i=1}^{3}(n_i - n'_i)$. To unravel such degeneracies one needs a set of six mutually commuting operators. One such set was introduced in Paper IV which can be easily expressed in terms of the T operators. Here we introduce a second and a more elegant alternative, by exploring two possible subgroups of the dynamical group.
(i) The operator

$$\mathscr{H} = T_i^\dagger T_i + \frac{3}{2}, \tag{14}$$

and its complex conjugate $\mathscr{H}^*$ form an abelian subgroup.
(ii) The operators

$$\begin{aligned}\mathscr{J}_i &= -i\epsilon_{ijk}T_j^\dagger T_k \\ &= \frac{1}{2}\epsilon_{ijk}\left\{\left(q_j p_k - \frac{\partial^2}{\partial q_j \partial p_k}\right) - i(p_j\frac{\partial}{\partial p_k} + q_j\frac{\partial}{\partial q_k})\right\},\end{aligned} \tag{15}$$

and their complex conjugates form a second subgroup. They have a $SO(4) \sim SO(3) \oplus SO(3)$ algebra. Thus,

$$[\mathscr{J}_i, \mathscr{J}_j] = i\epsilon_{ijk}\mathscr{J}_k, \tag{16a}$$

$$[\mathscr{J}_i^*, \mathscr{J}_j^*] = -i\epsilon_{ijk}\mathscr{J}_k^*, \tag{16b}$$

$$[\mathscr{J}_i, \mathscr{J}_j^*] = 0. \tag{16c}$$

Evidently $\mathscr{J}_i$ and $\mathscr{J}_i^*$ have angular momentum algebras with the Casimir operators $\mathscr{J}^2$ and $\mathscr{J}^{*2}$. Thus a complete set of mutually commuting operators may be written as

$$\{\mathscr{H}, \mathscr{H}^*, \mathscr{J}^2, \mathscr{J}^{*2}, \mathscr{J}_3, \mathscr{J}_3^*\}. \tag{17}$$

Liouville's operator in the notation of this section is $\mathscr{L} = \mathscr{H} - \mathscr{H}^*$. It, obviously, commute with the set of Eq. (17) and its eigensolutions will be given in terms of those of the commuting set.

5. Simultaneous eigensolutions of the commuting set

Let $f_{njm}^{n'j'm'}$ be the eigensolutions in question satisfying the relations

$$\mathscr{H} f_{njm}^{n'j'm'} = n f_{njm}^{n'j'm'}, \tag{18a}$$

$$\mathscr{H}^* f_{njm}^{n'j'm'} = n' f_{nkm}^{n'j'm'}, \tag{18b}$$

$$\mathscr{J}^2 f_{njm}^{n'j'm'} = j(j+1) f_{njm}^{n'j'm'}; j = \text{positive integer}, \tag{18c}$$

$$\mathscr{J}_3 f_{njm}^{n'j'm'} = m f_{njm}^{n'j'm'}; -j \le m \le j, \tag{18d}$$

$$= j'(j'+1) f_{njm}^{n'j'm'}; j' = \text{positive integer}, \tag{18e}$$

$$\mathscr{J}_3^* f_{njm}^{n'j'm'} = m' f_{njm}^{n'j'm'}; -j' \le m' \le j'. \tag{18f}$$

Equations (18a,b) are definitions for n and n'. Their allowed values will be discussed shortly. The limitations on j, m, j', and m' in Eqs. (18c,d) are dictated by the fact that $\mathscr{J}_i$ and $\mathscr{J}_i^*$ have the angular momentum algebras. It is worthwhile to mention that $J_i \equiv \mathscr{J}_i - \mathscr{J}_i^*$ is itself an angular momentum operator. Its eigennumbers are integer and range from $| j - j' |$ to $j + j'$ in integer steps. Similarly the eigenvalues of J_3 are $m - m'$.

We now proceed to the calculation of the eigensolutions $f_{njm}^{n'j'm'}$. The twelve operators T_i and $T^*{}_i$ and their hermitian adjoint together with the identity element, E, form a closed Lie algebra. The group associated with it is commonly referred to as the Heisenberg group and is designated by $N(6)$. Members of $N(6)$ commute with $\mathscr{D}$ but not with $\mathscr{L}$. The following commutators show the ladder properties of the T-operators.

$$[\mathscr{H}, T_\pm] = -T_\pm, \qquad [\mathscr{H}^*, T^*{}_\pm] = -T^*{}_\pm, \tag{19a}$$

$$[\mathscr{H}, T_\pm^\dagger] = T_\pm^\dagger, \qquad [\mathscr{H}^*, T^*{}_\pm^\dagger] = T^*{}_\pm^\dagger, \tag{19b}$$

$$[\mathscr{H}, T_\pm^*] = 0, \qquad [\mathscr{H}^*, T_\pm] = 0, \tag{19c}$$

$$[\mathscr{H}, T^*{}_\pm^\dagger] = 0, \qquad [\mathscr{H}^*, T_\pm^\dagger] = 0, \tag{19d}$$

$$[\mathscr{H}, T^{\dagger 2}] = 2T^{\dagger 2}, \qquad [\mathscr{H}^*, T^{*\dagger 2}] = 2T^{*\dagger 2}, \tag{19e}$$

$$[\mathscr{H}, T^{*\dagger 2}] = 0, \qquad [\mathscr{H}^*, T^{\dagger 2}] = 0, \tag{19f}$$

where $T^2 = T_i T_i$ and $T_\pm = T_1 \pm iT_2$, with similar relations for T^*'s. Also it is easy to see that the operator T_i is a class-T operator for $\mathscr{J}_i$ but commutes with $\mathscr{J}_i^*$. While $T^*{}_i$ is a class-T operator for $\mathscr{J}_i^*$ but commutes with $\mathscr{J}_i$. This fact and Eqs. (19), by standard arguments, lead to

$$T_+^\dagger f_{njm}^{n'j'm'} \propto f_{n+1,j+1,m+1}^{n'j'm'}, \tag{20a}$$

$$T_+^{*\dagger} f_{njm}^{n'j'm'} \propto f_{njm}^{n'+1,j'+1,m'+1}. \tag{20b}$$

Thus, using Eqs. (19)–(20) one may write $f_{njm}^{n'j'm'}$ as

$$\begin{aligned} f_{njm}^{n'j'm'} &\propto \mathscr{J}_+^{*(j'-m')} T_+^{*\dagger j'} (T^{*\dagger 2})^{\frac{n'-j'}{2}} \\ &\times \mathscr{J}_-^{(j-m)} T_+^{\dagger j} (T^{\dagger 2})^{\frac{n-j}{2}} f_{000}^{000}, \end{aligned} \tag{21}$$

where f_{000}^{000} is a generating function and happens to be the lowest eigenfunction for the positive definite operator $\mathscr{H}$. Since the power of $T^{\dagger 2}$ must be integer, n and j must both be even or odd integers and $n \ge j$. Similar restrictions hold for n' and j'. To find f_{000}^{000}, we note that $\mathscr{H}$ is positive definite. For, $\langle \mathscr{H} \rangle = \langle f \mid T_i^\dagger T_i + \frac{3}{2} \mid f \rangle = \langle T_i f \mid T_i f \rangle + \frac{3}{2}\langle f \mid f \rangle \ge$. Since T_i is a lowering ladder for $\mathscr{H}$ (see Eq. (19a)) one must have

$$T_i f_{000}^{000} = \frac{e^{it}}{\sqrt{2}}\left(p_i + \frac{\partial}{\partial p_i} - iq_i - i\frac{\partial}{\partial q_i}\right) f_{000}^{000} = 0. \tag{22a}$$

The most general solution of this equation is

$$f_{000}^{000}(\mathbf{q},\mathbf{p}) = \mathrm{e}^{-E}; \qquad E = \frac{1}{2}(p^2+q^2). \tag{22b}$$

The eigensolutions of Eq. (21) are also the simultaneous eigensolutions of $\mathscr{L} = \mathscr{H} - \mathscr{H}^*$ and $\mathscr{F} = \mathscr{H} + \mathscr{H}^*$. Thus,

$$\mathscr{L} f_{njm}^{n'j'm'} = (n-n') f_{njm}^{n'j'm'}, \tag{23a}$$

$$\mathscr{F} f_{njm}^{n'j'm'} = (n+n') f_{njm}^{n'j'm'}. \tag{23b}$$

The $\mathscr{F}$ operator, first introduced in Paper II and elaborated on by Khalesi (1990), has an interesting interpretation. It is the sum of two Schrodinger like hamiltonian operator, one in configuration representation and the other in the momentum representation.

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