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Localized solutions of the linearized gravitational field equations in free space

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Abstract. We consider the equations of general relativity in free space in the linear approximation. Non-stationary moving solutions for these equations, which are localized and have finite energy, are explicitly constructed. The energy of the localized wave is shown to be proportional to its internal frequency and can represent a massive quantum particle 'made' of gravitational energy.

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1. Introduction

Recently, moving localized solutions of massless scalar and electromagnetic fields were constructed [1, 2]. These solutions, called wavelets, move without spreading with a dispersion relation corresponding to that of a 'massive' quantum particle with group velocity v and phase velocity u satisfying $uv = c^2$. The wavelet is characterized by an internal frequency Ω such that its central peak is within a distance c/Ω and the corresponding mass is $\hbar\Omega/c$. For a single sharp frequency Ω_0 the wavelet is stable. However, to have a finite total energy a frequency distribution $f(\Omega)$ around Ω_0 of width Δ must be taken. In this case the lifetime of the state is of the order of $1/\Delta$. The integrated energy and momentum are proportional to the frequency and wavenumber of the moving solution, respectively.

These solutions have been interpreted in a 'quantum theory of single events'§ in such a way that the standard quantum mechanics emerges after averaging over the wavelets with different initial conditions. Furthermore, for $\Omega \to \infty$, the wavelets degenerate into moving delta functions, and we obtain as a consequence a limit to classical mechanics.

The purpose of this paper is to explicitly construct the wavelet solutions for the linearized gravitational field equations.

2. Field equations

In the linear approximation to general relativity one expresses the metric $g_{\mu\nu}$ as $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, where $\eta_{\mu\nu}$ is the Minkowski metricf with signature $(1, -1, -1, -1, -1)$ and $h_{\mu\nu}$

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§ For a review see [3].

is a correction. Likewise we have that $g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu}$. In this approximation Einstein's equations of general relativity become [5]

$$- \tfrac{1}{2}\partial^\gamma \partial_\gamma \bar{h}_{\alpha\beta} + \partial^\gamma \partial_{(\beta} \bar{h}_{a)\gamma} - \tfrac{1}{2}\eta_{\alpha\beta}\partial^\gamma \partial^\epsilon \bar{h}_{\gamma\epsilon} = \frac{8\pi k}{c^4} T_{\alpha\beta} \tag{1}$$

where $T_{\alpha\beta}$ is the energy–momentum tensor, k is the universal constant of gravitation, and $\bar{h}_{\alpha\beta}$ is defined as

$$\bar{h}_{\alpha\beta} = h_{\alpha\beta} - \tfrac{1}{2}\eta_{\alpha\beta} \tag{2}$$

with $h_\alpha{}^\alpha = h$. Note that we can invert this last relation as follows

$$\bar{h}_{\alpha\beta} = h_{\alpha\beta} - \tfrac{1}{2}\eta_{\alpha\beta}\bar{h} \tag{3}$$

with $\bar{h}^\mu_\mu = h$.

Let us now transform the metric as follows

$$h_{\alpha\beta} \to h_{\alpha\beta} + \partial_\alpha \xi_\beta + \partial_\beta \xi_\alpha \,. \tag{4}$$

The new metric is equivalent to the original if the vector field ξ_α generates an infinitesimal diffeomorphism in the spacetime [4].

Under this transformation we have

$$\bar{h}_{\alpha\beta} \to \bar{\bar{h}}_{\alpha\beta} = \bar{h}_{\alpha\beta} + \partial_\beta \xi_\alpha + \partial_\alpha \xi_\beta - \eta_{\alpha\beta}\partial^\delta \xi_\delta \tag{5}$$

and

$$\partial^\beta \bar{\bar{h}}_{\alpha\beta} = \partial^\beta \bar{h}_{\alpha\beta} + \partial^\beta \partial_\beta \xi_\alpha \,. \tag{6}$$

Choosing ξ_α such that $\partial_\beta \partial^\beta \xi_\alpha = -\partial^\beta \bar{h}_{\alpha\beta}$ one obtains $\partial_\beta \bar{\bar{h}}_\alpha \beta = 0$. Therefore assuming that $\partial^\beta \bar{h}_{\alpha\beta} = 0$, Einstein's equations become, with $T_{\mu\nu} = 0$, the free space wave equation $\partial^\alpha \partial_\alpha \bar{h}_{\beta\gamma}$.

3. Localized solutions

The localized solutions are first constructed in the rest frame of the lump at some point x_0 of the space. The solution in an arbitrary inertial frame will be obtained by a Lorentz transformation.

We shall assume that $\bar{h}^{1\alpha} = \bar{h}^{2\alpha} = 0$ and that $\bar{h}^{00}$, $\bar{h}^{03}$, $\bar{h}^{30}$, $\bar{h}^{33}$ are the only non-zero components. Thus, we have the metric tensors

$$(\bar{h})^\mu{}_\nu = \begin{pmatrix} \bar{h}^0{}_0 & \bar{h}^0{}_3 \\ \bar{h}^3{}_0 & \bar{h}^3{}_3 \end{pmatrix} \tag{7}$$

$$(h)^\mu{}_\nu = \begin{pmatrix} \tfrac{1}{2}(\bar{h}^0{}_3 - \bar{h}^3{}_3) & \bar{h}^0{}_3 \\ \bar{h}^3{}_0 & -\tfrac{1}{2}(\bar{h}^0{}_3 - \bar{h}^3{}_3) \end{pmatrix} . \tag{8}$$

Now we find solutions for the wave equation having azimuthal symmetry. We assume the following oscillatory form for the components of the metric

$$\bar{h}^{\alpha 0} \equiv \bar{h}^0 = \lambda \sum_{\ell} c^0_{(\ell)} P_\ell(\cos\theta) \int j_\ell(\xi) f(\Omega)\, \mathrm{e}^{\mathrm{i}\Omega t} \sqrt{\frac{\Omega}{c}}\, \mathrm{d}\Omega$$
$$\bar{h}^{\alpha 3} \equiv \bar{h}^3 = \lambda \sum_{\ell} c^3_{(\ell)} P_\ell(\cos\theta) \int j_\ell(\xi) f(\Omega)\, \mathrm{e}^{\mathrm{i}\Omega t} \sqrt{\frac{\Omega}{c}}\, \mathrm{d}\Omega \tag{9}$$

where $\xi = (\Omega/c)r$ and λ is a constant with dimensions of time$\sqrt{\text{length}}$. It is implicit in the form of equation (9) that it refers to a particular inertial observer, namely, the one that observes a non-moving centre of localization. The metric $\bar{h}_{\alpha\beta}$ is dimensionaless and j_ℓ is a Bessel function

$$j_\ell(\xi) = \sqrt{\frac{\pi}{2\xi}}\, J_{\ell+1/2}(\xi). \tag{10}$$

These solutions are required to satisfy the gauge condition

$$\partial^\alpha \bar{h}_{\alpha\beta} = \frac{1}{c}\frac{\partial}{\partial t}\bar{h}^0 + \frac{\partial}{\partial z}\bar{h}^3 = 0. \tag{11}$$

Substitution into the gauge condition gives, after using the identities

$$(1-x^2)\frac{\mathrm{d}}{\mathrm{d}x}P_\ell(x) = \ell P_{\ell-1}(x) - \ell x P_\ell(x) = \frac{\ell(\ell+1)}{2\ell+1}\{P_{\ell-1} - P_{\ell+1}\}$$
$$x P_\ell = \frac{1}{2\ell+1}\{\ell P_{\ell-1} + (\ell+1)P_{\ell+1}\}$$
$$j_\ell(x) = \frac{x}{2\ell+1}\{j_{\ell-1}(x) + j_{\ell+1}(x)\}$$
$$\frac{\mathrm{d}j_\ell(x)}{\mathrm{d}x} = \frac{1}{2\ell+1}\{\ell j_{\ell-1}(x) - (\ell+1)j_{\ell+1}(x)\} \tag{12}$$

the expressions

$$\frac{1}{c}\frac{\partial}{\partial t}\bar{h}^0 = \lambda \sum_{\ell=0} \mathrm{i}c^0{}_{(\ell)} P_\ell j_{(\ell)}$$
$$\frac{\partial}{\partial z}\bar{h}^3 = \tfrac{1}{3}\lambda c^3{}_{\ell+1} P_0 j_{(0)} + \lambda \sum_{\ell=1} \left\{\frac{c^3{}_{\ell+1}(\ell+1)}{2\ell+1} - \frac{\ell c^3{}_{\ell-1}}{2\ell-1}\right\} P_\ell j_{(\ell)} \tag{13}$$

where

$$j_{(i)} \equiv \int \frac{\Omega}{c}\, j_i(\xi) f(\Omega)\, \mathrm{e}^{\mathrm{i}\Omega t} \sqrt{\frac{\Omega}{c}}\, \mathrm{d}\Omega.$$

From these derivatives one obtains the recurrence relations

$$\begin{aligned} &\tfrac{1}{3}c^3_{(\ell)} + \mathrm{i}c^0_{(0)} = 0 \\ &\mathrm{i}c^0_{(\ell)} + \frac{\ell+1}{2\ell+3}\,c^3_{(\ell+1)} - \frac{\ell}{2\ell-1}\,c^3_{(\ell-1)} = 0 \end{aligned} \tag{14}$$

for $\ell \geqslant 1$. The simplest solutions of these equations are

$$\bar{h}^0 = \lambda P_1 j_{(1)} \qquad \bar{h}^3 = \mathrm{i} j_{(0)} \tag{15}$$

and

$$\bar{h}^0 = \tfrac{1}{3}\mathrm{i}\lambda\{j_{(0)} - 2P_2 j_{(2)}\} \qquad \bar{h}^3 = \lambda P_1 j_{(1)} \tag{16}$$

where now

$$j_{(i)} \equiv \int j_i(\xi) f(\Omega)\, \mathrm{e}^{\mathrm{i}\Omega t}\sqrt{\frac{\Omega}{c}}\,\mathrm{d}\Omega$$

i.e. without the factor Ω/c inside the integral.

Thus our solution consists of

$$(\bar{h})^{\mu\nu} = \lambda \begin{pmatrix} \frac{1}{3}\mathrm{i}\{j_0 - 2P_2 j_2\} & P_1 j_1 \\ P_1 j_1 & \mathrm{i} j_0 \end{pmatrix} \qquad \bar{h} = -\tfrac{2}{3}\lambda\mathrm{i}\{j_0 + P_2 j_2\} \tag{17}$$

and

$$(h)^{\mu\nu} = \lambda \begin{pmatrix} \frac{1}{3}\mathrm{i}\{2j_0 - P_2 j_2\} & P_1 j_1 \\ P_2 j_1 & \frac{1}{3}\mathrm{i}\{2j_0 - P_2 j_2\} \end{pmatrix}. \tag{18}$$

4. Energy of the localized solution

For the purpose of obtaining the energy of the localized solution let us compute the zero–zero component of the energy–momentum pseudo-tensor for the gravitational field found by Landau and Lifshitz [5].

In the linear approximation it becomes

$$\begin{aligned} t_{00} = \frac{c^4}{16\pi k}\, \big\{ & h^{00}{}_{,l}h^{lm}{}_{,m} - h^{0l}{}_{,l}h^{0m}{}_{,m} + \tfrac{1}{2}\eta_{lm}h^{nl}{}_{,p}h^{pm}{}_{,n} - \eta^{0l}\eta_{mn}h^{0n}{}_{,p}h^{mp}{}_{,l} \\ & - \eta^{0l}\eta_{mn}h^{0n}{}_{,p}h^{mp}{}_{,l} + \eta^{np}\eta_{lm}h^{0l}{}_{,n}h^{mp}{}_{,p} \\ & + \tfrac{1}{8}(2\eta^{0l}\eta^{0m} - \eta^{lm})(2\eta_{pq}\eta_{qr} - \eta_{pq}\eta_{nr})h^{nr}{}_{,l}h^{pq}{}_{,m} \big\}\,. \end{aligned} \tag{19}$$

This expression was obtained by keeping only terms of first order in $h_{\alpha\beta}$ and using

$$g = \det(g_{\alpha\beta}) = \det(\eta_{\alpha\beta} + h_{\alpha\beta}) = -1. \tag{20}$$

Now, setting $h^{00} = h^{33}$ we obtain

$$t^{00} = \frac{c^4}{32\pi k}\{(h^{03}{}_{,1})^2 + (h^{03}{}_{,2})^2 - (h^{00}{}_{,1})^2 - (h^{00}{}_{,2})^2\}. \tag{21}$$

The following derivatives are needed to calculate the energy (with $k = 1, 2, 3$)

$$\begin{aligned} h^{00}{}_{,k} &= \tfrac{1}{3}\mathrm{i}\lambda\left\{-2j_{(1)}\frac{\Omega}{c}(1 + \tfrac{1}{5}P_2)\partial_k r + j_{(2)}\frac{2\cos\theta}{\sin^2\theta}(P_2 - 1)\partial_k\cos\theta + P_2 J_{(3)}\frac{3\Omega}{5c}\partial_k r\right\} \\ h^{03}{}_{,k} &= \lambda\left\{j_{(0)}\frac{\Omega\cos\theta}{3c}\partial_k r + j_{(1)}\partial_k\cos\theta - j_{(2)}\frac{2\Omega}{3c}\cos\theta\,\partial_k r\right\}. \end{aligned} \tag{22}$$

Recall that in these expressions

$$\frac{\Omega}{c}j_{(i)} \equiv \int\frac{\Omega}{c}j_i x i) f(\Omega)\,\mathrm{e}^{\mathrm{i}\Omega t}\sqrt{\frac{\Omega}{c}}\,\mathrm{d}\Omega.$$

Since for our solution

$$E = \int t^{00}\,\mathrm{d}^3x = \frac{c^4}{16\pi k}\left\{\int(h^{03}{}_{,1})^2\,\mathrm{d}x - \int(h^{00}{}_{,1})^2\,\mathrm{d}x\right\} \tag{23}$$

we just need

$$\begin{aligned} \int(h^{03}{}_{,1})^2\,\mathrm{d}^3x = \lambda^2\Bigg\{&(\alpha_{00} + \alpha_{22})\int\Omega|f(\Omega)|^2\,\mathrm{d}\Omega \\ &+ \alpha_{11}\int j_1 j_1'\sqrt{\frac{\Omega\Omega'}{c^2}}\,ff'\,\mathrm{e}^{\mathrm{i}(\Omega-\Omega')t}\,\mathrm{d}t\,\mathrm{d}\Omega\,\mathrm{d}\Omega' \\ &+ \int\{\alpha_{01}j_0 j_1' + \alpha_{12}j_1 j_2'\}\left\{\frac{\Omega}{c} + \frac{\Omega'}{c}\right\}\sqrt{\frac{\Omega\Omega'}{c}} \\ &\times ff'\,\mathrm{e}^{\mathrm{i}(\Omega-\Omega')t}r\,\mathrm{d}r\,\mathrm{d}\Omega\,\mathrm{d}\Omega'\Bigg\} \end{aligned} \tag{24}$$

and

$$\begin{aligned} -\int(h^{00}{}_{,1})^2\,\mathrm{d}^3x = \lambda^2\Bigg\{&(\beta_{11} + \beta_{33})\int\Omega|f(\Omega)|^2\,\mathrm{d}\Omega \\ &+ \beta_{22}\int j_2 j_2'\sqrt{\frac{\Omega\Omega'}{c}}\,ff'\,\mathrm{e}^{\mathrm{i}(\Omega-\Omega')t}\,\mathrm{d}r\,\mathrm{d}\Omega\,\mathrm{d}\Omega' \\ &+ \int\{\beta_{12}j_1 j_2' + \beta_{23}j_2 j_3'\}\left\{\frac{\Omega}{c} + \frac{\Omega'}{c}\right\}\sqrt{\frac{\Omega\Omega'}{c^2}} \\ &\times ff'\,\mathrm{e}^{\mathrm{i}(\Omega-\Omega')t}r\,\mathrm{d}r\,\mathrm{d}\Omega\,\mathrm{d}\Omega'\Bigg\} \end{aligned} \tag{25}$$

where

$$\alpha_{00} = \frac{4\pi}{9*15} \quad \alpha_{22} = \frac{16\pi}{9*15} \quad \alpha_{11} = \frac{4\pi}{9*15} \quad \alpha_{01} = -\frac{8\pi}{6*15} \quad \alpha_{12} = \frac{8\pi}{3*15} \tag{26}$$

and

$$\begin{aligned} \beta_{11} &= -\frac{8*287.55\pi}{9*420} \qquad \beta_{22} = -\frac{2\pi}{35} \qquad \beta_{33} = -\frac{8\pi}{21*25} \\ \beta_{12} &= -\frac{108\pi}{63*50} \qquad \beta_{23} = -\frac{12\pi}{25*63}. \end{aligned} \tag{27}$$

In expressions (24) and (25) we have used the notations $f = f(\Omega)$, $f' = f(\Omega')$, $j_1 = j_1(\xi)$, $j_1' = j_1(\xi')$, etc.

Hence the total energy takes the final form

$$\begin{aligned} E &= \int t^{00}\,\mathrm{d}^3z \\ &= \frac{c^4\lambda^2}{16\pi k}\left\{(\alpha_{00}+\alpha_{22}+\beta_{11}+\beta_{33})\int \Omega|f(\Omega)|^2\,\mathrm{d}\Omega\right. \\ &\quad + \alpha_{11}\int \frac{J_{3/2}J'_{3/2}}{r} ff'\,\mathrm{e}^{\mathrm{i}(\Omega-\Omega')t}\,\mathrm{d}\Omega\,\mathrm{d}\Omega'\,\mathrm{d}r \\ &\quad + \beta_{22}\int \frac{J_{5/2}J'_{5/2}}{r} ff'\,\mathrm{e}^{\mathrm{i}(\Omega-\Omega')t}\,\mathrm{d}\Omega\,\mathrm{d}\Omega'\,\mathrm{d}r \\ &\quad + \int \{\alpha_{01}J_{1/2}J_{3/2} + (\alpha_{12}+\beta_{12})J_{3/2}J'_{5/2} + \beta_{23}J_{5/2}J'_{7/2}\}\left\{\frac{\Omega}{c}+\frac{\Omega}{c}\right\} \\ &\quad \left.\times\, ff'\,\mathrm{e}^{\mathrm{i}(\Omega-\Omega')t}\,\mathrm{d}\Omega\,\mathrm{d}\Omega'\,\mathrm{d}r\right\} \end{aligned} \tag{28}$$

or, after integration of r from zero to infinity,

$$\begin{aligned} E = \frac{c^4\lambda^2}{16\pi k}&\left\{\gamma_1\int_0^\infty \Omega|f(\Omega)|^2\,\mathrm{d}\Omega\right. \\ &+ \gamma_2\int \theta(\Omega-\Omega')\left\{\left(\frac{\Omega'}{\Omega}\right)^{3/2}+\left(\frac{\Omega'}{\Omega}\right)^{5/2}\right\} ff'2\,\mathrm{Re}\,(\mathrm{e}^{\mathrm{i}(\Omega-\Omega')t})\,\mathrm{d}\Omega\,\mathrm{d}\Omega' \\ &+ \int \theta(\Omega-\Omega')ff'\,\mathrm{e}^{\mathrm{i}(\Omega-\Omega')t} \\ &\left.\times\left\{\alpha_{01}\left(\frac{\Omega}{\Omega'}\right)^{1/2}+\gamma_3\left(\frac{\Omega}{\Omega'}\right)^{3/2}+\gamma_4\left(\frac{\Omega}{\Omega'}\right)^{5/2}+\beta_{23}\left(\frac{\Omega}{\Omega'}\right)^{7/2}\right\}\mathrm{d}\Omega\,\mathrm{d}\Omega'\right\} \end{aligned} \tag{29}$$

which is our final result. Here

$$\begin{aligned} \gamma_1 &= \alpha_{00}+\alpha_{22}+\beta_{11}+\beta_{33} \qquad \gamma_2 = \tfrac{1}{3}\alpha_{11}+\tfrac{1}{5}\beta_{22} \\ \gamma_3 &= \alpha_{01}+\alpha_{12}+\beta_{12} \qquad \gamma_4 = \alpha_{12}+\beta_{12}+\beta_{23}. \end{aligned} \tag{30}$$

5. Concluding remarks

If the frequency function $f(\Omega)$ is sufficiently localized, the second and third integrals in equation (29) could be negligible and the energy of the 'particle' becomes proportional to the average of the internal frequency. It is important to point out that without the superposition with the frequency function $f(\Omega)$, the energy of our solutions is infinite.

As in the case of electrodynamics and the massless scalar theories, when the frequency approaches infinity the localized solution behaves like a point particle. Finally, in any inertial frame the solution reads

$$h'_{\mu\nu}(r_\mu, \beta) = \Lambda^{\mu'}_{\mu} \Lambda^{\nu'}_{\nu} h_{\mu'\nu'}(r_\perp) \tag{31}$$

where

$$\begin{aligned} & r_\perp^2 = (r^\mu_\mu)^2 - r_\mu r^\mu \qquad n_\mu = \gamma(1, \beta) \\ & r_\perp = (r^2 + \beta^2\gamma^2 t^2 - 2\gamma^2(\tilde{\beta} \cdot \tilde{r}) + \gamma^2(\tilde{\beta} \cdot \tilde{r})^2)^{1/2} \\ & \tilde{r} = \tilde{x} - \tilde{a}. \end{aligned} \tag{32}$$

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