Integrals and static solutions of general relativistic Liouville's equation in post Newtonian approximation

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1

Abstract The post-Newtonian approximation of general relativistic Liouville's equation is presented. Two integrals of it, generalizations of the classical energy and angular momentum, are obtained. Polytropic models are constructed as an application.

Key word:relativistic systems:static structures; methods: numerical

1   Introduction

Solutions of general relativistic Liouville's equation (grl) in a prescribed space-time have been considered by some investigators. Most authors have sought its solutions as functions of the constants of motion, generated by Killing vectors of the space-time in question. See for example Ehlers (1971), Ray and Zimmerman (1977), Mansouri and Rakei (1988), Ellis, Matraverse and Treciokas (1983), Maharaj and Maartens (1985, 1987), Maharaj (1989), and Dehghani and Rezania (1996).

In applications to self gravitating stars and stellar systems, however, one should combine Einstein's field equations and grl. The resulting nonlinear equations can be solved in certain approximations. Two such methods are available; the *post-Newtonian (pn) approximation* and the *weak-field* one. In this paper we adopt the first approach to study a self gravitating system imbeded in an otherwise flat space-time. In sect. 2, we derive the grl in the pn approximation. In sect. 3 we seek static solutions of the post-Newtonian Liouville's equation (pnl). We find two integrals of pnl that are the pn generalizations of the energy and angular momentum integrals of the classical Liouville's equation. Post-Newtonian polytropes, as simultaneous solutions of pnl and Einsteins equation, are discussed and calculated in sect. 4. Section 5 is devoted to concluding remarks.

The main objective of this paper, however, is to set the stage for the second paper in this series (Sobouti and Rezania, 1998). There, we study a class of non static oscillatory solutions of pnl, different from the conventional p and g modes of the system. They are associated with oscillations in the space-time metric, without disturbing the classical equilibrium of the system. In this respect they might be akin to the so called *gravitational wave* modes that some authors have advocated to exist in relativistic systems. See, for example, Andersson, Kokkotas and Schutz (1995), Baumgarte and Schmidt (1993), Detweiler and Lindblom (1983, 1985), Kojima (1988), Kokkotas and Schutz (1986, 1992), Leaver (1985), Leins, Nollert and Soffel (1993),

Lindblom, Mendell and Ipser (1997), and Nollert and Schmidt (1992).

2 Liouville's equation in post-Newtonian approximation

The one particle distribution function of a gas of collisionless particles with identical masse m, in the restricted seven dimensional phase space

$$P(m): \quad g_{\mu\nu}U^\mu U^\nu = -1 \tag{1}$$

satisfies *grl*:

$$\mathcal{L}_U f = (U^\mu \frac{\partial}{\partial x^\mu} - \Gamma^i_{\mu\nu} U^\mu U^\nu \frac{\partial}{\partial U^i}) f(x^\mu, U^i) = 0, \tag{2}$$

where (x^μ, U^i) is the set of configuration and velocity coordinates in $P(m)$, $f(x^\mu, U^i)$ is a distribution function, $\mathcal{L}_U$ is Liouville's operator in the (x^μ, U^i) coordinates, and $\Gamma^i_{\mu\nu}$ are Christoffel's symbols. Greek indices run from 0 to 3 and Latin indices from 1 to 3. We use the convention $c = 1$ except in numerical calculations of section 4. The four-velocity of the particle and its classical velocity are related as

$$U^\mu = U^0 v^\mu; \quad v^\mu = (1, v^i = dx^i/dt), \tag{3}$$

where $U^0(x^\mu, v^i)$ is to be determined from Eq. (1). In the *pn* approximation, we need an expansion of $\mathcal{L}_U$ up to order $\bar{v}^4$, where $\bar{v}$ is a typical Newtonian speed. To achieve this goal we transform (x^μ, U^i) to (x^μ, v^i). Liouville's operator transforms as

$$\mathcal{L}_U = U^0 v^\mu (\frac{\partial}{\partial x^\mu} + \frac{\partial v^j}{\partial x^\mu} \frac{\partial}{\partial v^j}) - \Gamma^i_{\mu\nu} U^{0^2} v^\mu v^\nu \frac{\partial v^j}{\partial U^i} \frac{\partial}{\partial v^j}, \tag{4}$$

where $\partial v^j/\partial x^\mu$ and $\partial v^j/\partial U^i$ are determined from the inverse of the transformation matrix (see appendix A). Thus,

$$\frac{\partial v^j}{\partial x^\mu} = -\frac{U^0}{2Q} v^j \frac{\partial g_{\alpha\beta}}{\partial x^\mu} v^\alpha v^\beta, \tag{5a}$$

$$\frac{\partial v^j}{\partial U^i} = \frac{1}{Q} v^j (g_{0i} + g_{ik} v^k); \qquad \text{for } i \neq j, \tag{5b}$$

$$= -\frac{1}{Q} (U^{0^{-2}} + \sum_{k \neq i} v^k (g_{0k} + g_{kl} v^l)); \quad \text{for } i = j,$$

where

$$Q = U^0 (g_{00} + g_{0l} v^l). \tag{5c}$$

3

Using Eqs.(5) in Eq.(4), one finds

$$\mathcal{L}_U f = U^0 \mathcal{L}_v f = 0, \tag{6a}$$

or

$$\mathcal{L}_v f(x^\mu, v^i) = 0, \tag{6b}$$

where

$$\mathcal{L}_v = v^\mu \left(\frac{\partial}{\partial x^\mu} - \frac{U^0}{2Q} v^j \frac{\partial g_{\alpha\beta}}{\partial x^\mu} v^\alpha v^\beta \frac{\partial}{\partial v^j} \right) - \Gamma^i_{\mu\nu} v^\mu v^\nu \left\{ \sum_{j \neq i} \frac{1}{Q} v^j (g_{0i} + g_{ik} v^k) \frac{\partial}{\partial v^j} \right.$$
$$\left. - \frac{1}{Q} (U^{0^{-2}} + \sum_{k \neq i} v^k (g_{0k} + g_{kl} v^l)) \frac{\partial}{\partial v^i} \right\}, \tag{6c}$$

We note that the post-Newtonian hydrodynamic equations are obtained from integrations of Eq. (6a) over the **v**-space rather than Eq. (6b) (see appendix B). We expand $\mathcal{L}_v$ up to the order $\bar{v}^4$. For this purpose, we need expansions of Einstein's field equations, the metric tensor, and the affine connections up to various orders. Einstein's field equation with harmonic coordinates condition, $g^{\mu\nu} \Gamma^\lambda_{\mu\nu} = 0$, yields (see Weinberg 1972):

$$\nabla^2 {}^2g_{00} = -8\pi G \, {}^0T^{00}, \tag{7a}$$

$$\nabla^2 {}^4g_{00} = \frac{\partial^2 \, {}^2g_{00}}{\partial t^2} + {}^2g_{ij} \frac{\partial^2 \, {}^2g_{00}}{\partial x^i \partial x^j} - \left(\frac{\partial \, {}^2g_{00}}{\partial x^i} \right) \left(\frac{\partial \, {}^2g_{00}}{\partial x^i} \right)$$
$$- 8\pi G ({}^2T^{00} - 2 \, {}^2g_{00} \, {}^0T^{00} + {}^2T^{ii}), \tag{7b}$$

$$\nabla^2 {}^3g_{i0} = 16\pi G \, {}^1T^{i0}, \tag{7c}$$

$$\nabla^2 {}^2g_{ij} = -8\pi G \delta_{ij} \, {}^0T^{00}. \tag{7d}$$

The symbols ${}^n g_{\mu\nu}$ and ${}^n T^{\mu\nu}$ denote the nth order terms in $\bar{v}$ in the metric and in the energy-momentum tensors, respectively. Solutions of these equations are

$$ {}^2g_{00} = -2\phi, \tag{8a}$$

$$ {}^2g_{ij} = -2\delta_{ij}\phi, \tag{8b}$$

$$ {}^3g_{i0} = \eta_i, \tag{8c}$$

$$ {}^4g_{00} = -2\phi^2 - 2\psi, \tag{8d}$$

where

$$\phi(\mathbf{x}, t) = -G \int \frac{{}^0T^{00}(\mathbf{x}', t)}{|\mathbf{x} - \mathbf{x}'|} d^3 x', \tag{9a}$$

4

$$\eta^i(\mathbf{x}, t) = -4G \int \frac{{}^1T^{i0}(\mathbf{x}', t)}{|\mathbf{x} - \mathbf{x}'|} d^3x', \tag{9b}$$

$$\psi(\mathbf{x}, t) = -\int \frac{d^3x'}{|\mathbf{x} - \mathbf{x}'|} \left(\frac{1}{4\pi} \frac{\partial^2 \phi(\mathbf{x}', t)}{\partial t^2} + G\, {}^2T^{00}(\mathbf{x}', t) \right.$$
$$\left. + G\, {}^2T^{ii}(\mathbf{x}', t) \right), \tag{9d}$$

where bold characters denote the three vectors. Substituting Eqs. (8) and (9) in (6c) gives

$$
\begin{aligned}
\mathcal{L}_v &= \mathcal{L}^{cl} + \mathcal{L}^{pn} \\
&= \frac{\partial}{\partial t} + v^i \frac{\partial}{\partial x^i} - \frac{\partial \phi}{\partial x^i} \frac{\partial}{\partial v^i} \\
&\quad - \left[(4\phi + \mathbf{v}^2) \frac{\partial \phi}{\partial x^i} - \frac{\partial \phi}{\partial x^j} v^i v^j - v^i \frac{\partial \phi}{\partial t} + \frac{\partial \psi}{\partial x^i} \right. \\
&\quad \left. + \left(\frac{\partial \eta_i}{\partial x^j} - \frac{\partial \eta_j}{\partial x^i} \right) v^j + \frac{\partial \eta_i}{\partial t} \right] \frac{\partial}{\partial v^i}
\end{aligned}
\tag{10}$$

where $\mathcal{L}^{cl}$ and $\mathcal{L}^{pn}$ are the classical Liouville operator and its post-Newtonian correction, respectively. Equation (6b) for the distribution function $f(x^\mu, v^i)$ becomes

$$(\mathcal{L}^{cl} + \mathcal{L}^{pn}) f(x^\mu, v^i) = 0. \tag{11}$$

The three scalar and vector potentials ϕ, ψ and $\boldsymbol{\eta}$ can now be given in terms of the distribution function. The energy-momentum tensor in terms of $f(x^\mu, U^i)$ is

$$T^{\mu\nu}(x^\lambda) = \int \frac{U^\mu U^\nu}{U^0} f(x^\lambda, U^i) \sqrt{-g}\, d^3U, \tag{12}$$

where $g = det(g_{\mu\nu})$. For various orders of $T^{\mu\nu}$ one finds

$$ {}^0T^{00}(x^\lambda) = \int f(x^\lambda, v^i) d^3v, \tag{13a}$$

$$ {}^2T^{00}(x^\lambda) = \int \left(\frac{1}{2} v^2 + \phi(x^\lambda) \right) f(x^\lambda, v^i) d^3v, \tag{13b}$$

$$ {}^2T^{ij}(x^\lambda) = \int v^i v^j f(x^\lambda, v^i) d^3v, \tag{13c}$$

$$ {}^1T^{0i}(x^\lambda) = \int v^i f(x^\lambda, v^i) d^3v. \tag{13d}$$

Substituting Eqs. (13) in (9) gives

$$\phi(\mathbf{x}, t) = -G \int \frac{f(\mathbf{x}', t, \mathbf{v}')}{|\mathbf{x} - \mathbf{x}'|} d\Gamma', \tag{14a}$$

$$\boldsymbol{\eta}(\mathbf{x}, t) = -4G \int \frac{\mathbf{v}' f(\mathbf{x}', t, \mathbf{v}')}{|\mathbf{x} - \mathbf{x}'|} d\Gamma' \tag{14b}$$

5

$$\psi(\mathbf{x}, t) = \frac{G}{4\pi} \int \frac{\partial^2 f(\mathbf{x}'', t, \mathbf{v}'')/\partial t^2}{|\mathbf{x} - \mathbf{x}'||\mathbf{x}' - \mathbf{x}''|} d^3 x' d\Gamma''$$

$$- \frac{3}{2} G \int \frac{\mathbf{v}'^2 f(\mathbf{x}', t, \mathbf{v}')}{|\mathbf{x} - \mathbf{x}'|} d\Gamma'$$

$$+ G^2 \int \frac{f(\mathbf{x}', t, \mathbf{v}') f(\mathbf{x}'', t, \mathbf{v}'')}{|\mathbf{x} - \mathbf{x}'||\mathbf{x}' - \mathbf{x}''|} d\Gamma' d\Gamma'', \tag{14c}$$

where $d\Gamma = d^3 x d^3 v$. Equations (11) and (14) complete the *pn* order of Liouville's equation for self gravitating systems imbeded in a flat space-time.

3 Integrals of post-Newtonian Liouville's equation

In an equilibrium state $f(\mathbf{x}, \mathbf{v})$ is time-independent. Macroscopic velocities along with the vector potential $\boldsymbol{\eta}$ vanish. Equations (10) and (11) reduce to

$$(\mathcal{L}^{cl} + \mathcal{L}^{pn})f(\mathbf{x}, \mathbf{v}) = [(v^i \frac{\partial}{\partial x^i} - \frac{\partial \phi}{\partial x^i} \frac{\partial}{\partial v^i})$$

$$- (\frac{\partial \phi}{\partial x^i}(4\phi + v^2) - \frac{\partial \phi}{\partial x^j} v^i v^j + \frac{\partial \psi}{\partial x^i}) \frac{\partial}{\partial v^i}]f = 0, \tag{15}$$

One easily verifies that the following, a generalization of the classical energy integral, is a solution of Eq. (15)

$$E = \frac{1}{2} v^2 + \phi + 2\phi^2 + \psi + const. \tag{16}$$

Furthermore, if $\phi(\mathbf{x})$ and $\psi(\mathbf{x})$ are spherically symmetric, which actually is the case for an isolated system in an asymptotically flat space-time, the following generalization of angular momentum are also integrals of Eq. (15)

$$l_i = \varepsilon_{ijk} x^j v^k exp(-\phi), \tag{17}$$

where ε_{ijk} is the Levi-Cevita symbol. Static distribution functions maybe constructed as functions of E and even functions of l_i. The reason for restriction to even functions of l_i^{pn} is to insure the vanishing of η^i, the condition for validity of Eq. (15).

4 Polytropes in post-Newtonian approximation

As in classical polytropes we consider the distribution function for a polytrope of index n as

$$F_n(E) = \frac{\alpha_n}{4\pi\sqrt{2}}(-E)^{n-3/2}; \quad \text{for } E < 0,$$

$$= 0 \qquad\qquad \text{for } E > 0, \tag{18}$$

6

where α_n is a constant. By Eqs. (13) the corresponding orders of the energy-momentum tensor are

$$^0T_n^{00} = \alpha_n\beta_n(-U)^n, \tag{19a}$$

$$^2T_n^{00} = \alpha_n\beta_n\phi(-U)^n + \alpha_n\gamma_n(-U)^{n+1}, \tag{19b}$$

$$^2T_n^{ii} = \delta_{ij}\,^2T^{ij} = 2\alpha_n\gamma_n(-U)^{n+1}, \tag{19c}$$

$$^1T_n^{0i} = 0, \tag{19d}$$

where

$$\beta_n = \int_0^1 (1-\eta)^{n-3/2}\eta^{1/2}d\eta = \Gamma(3/2)\Gamma(n-1/2)/\Gamma(n+1), \tag{20}$$

$$\gamma_n = \int_0^1 (1-\eta)^{n-3/2}\eta^{3/2}d\eta = \Gamma(5/2)\Gamma(n-1/2)/\Gamma(n+2), \tag{21}$$

and $U = \phi + 2\phi^2 + \psi$ is the gravitational potential in pn order. It will be chosen zero at the surface of the stellar configuration. With this choice, the escape velocity $v_e = \sqrt{-2U}$ will mean escape to the boundary of the system rather than to infinity. Einstein's equations, Eqs. (7), (8) and (9), lead to

$$\nabla^2\phi = 4\pi G\,^0T^{00} = 4\pi G\alpha_n\beta_n(-U)^n, \tag{22}$$

$$\nabla^2\psi = 4\pi G(^2T^{00} +^2 T^{ii}) = 4\pi G\alpha_n\beta_n\phi(-U)^n$$
$$+12\pi G\alpha_n\gamma_n(-U)^{n+1}. \tag{23}$$

Expanding $(-U)^n$ as

$$(-U)^n = (-\phi)^n[1 + n(2\phi + \frac{\psi}{\phi}) + \cdots], \tag{24}$$

ans Substituting it in Eqs. (22) and (23) gives

$$\nabla^2\phi = 4\pi G\alpha_n\beta_n[(-\phi)^n - 2n(-\phi)^{n+1} - n(-\phi)^{n-1}\psi + \cdots], \tag{25}$$

$$\nabla^2\psi = 4\pi G\alpha_n\beta_n\{(3\frac{\gamma_n}{\beta_n} - 1)(-\phi)^{n+1}$$
$$-[3(n+1)\frac{\gamma_n}{\beta_n} - n][2(-\phi)^{n+2} + (-\phi)^n\psi] + \cdots\}, \tag{26}$$

These equations can be solved numerically by an iterative scheme. We introduce three dimesionless quantities

$$-\phi \equiv \lambda\theta, \tag{27a}$$

7

$$-\psi \equiv \lambda^2 \Theta, \tag{27b}$$

$$r \equiv a\zeta, \tag{27c}$$

where, in terms of ρ_c, the central density, $\lambda = (\rho_c/\alpha_n\beta_n)^{1/n}$ and $a^{-2} = 4\pi G\rho_c/\lambda$. Equations (25) and (26) in various iteration orders reduce to

$$\nabla^2_\zeta \theta_o + \theta_o^n = 0, \tag{28a}$$

$$\nabla^2_\zeta \Theta_o + (3\frac{\gamma_n}{\beta_n} - 1)\theta_o^{n+1} = 0, \tag{28b}$$

$$\nabla^2_\zeta \theta_1 + \theta_1^n = qn(2\theta_o^{n+1} - \theta_o^{n-1}\Theta_o), \tag{28c}$$

$$\nabla^2_\zeta \Theta_1 + (3\frac{\gamma_n}{\beta_n} - 1)\theta_1^{n+1} =$$
$$q[3(n+1)\frac{\gamma_n}{\beta_n} - n](2\theta_o^{n+2} - \theta_o^n\Theta_o), \tag{28d}$$

where $\nabla^2_\zeta = \frac{1}{\zeta^2}\frac{d}{d\zeta}(\zeta^2\frac{d}{d\zeta})$. The subscripts 0 and 1 refer to orders of the iteration. The dimensionless parameter q is defined as

$$q = \frac{4\pi G\rho_c a^2}{c^2} = \frac{R_s}{R}\frac{1}{2\zeta_1 \mid \theta_o'(\zeta_1) \mid}, \tag{29}$$

where R_s is the Schwarzschild radius, $R = a\zeta_1$ is the radius of system, ζ_1 is defined by $\theta_o(\zeta_1) = 0$ and c is the light speed. The order of magnitude of q varies from 10^{-5} for white dwarfs to 10^{-1} for neutron stars. For future reference, let us also note that

$$-U = \lambda[\theta_1 + q(\Theta_1 - 2\theta_1^2)]. \tag{30}$$

We use a forth-order Runge-Kutta method to find numerical solutions of the four coupled nonlinear differential Eqs. (28). At the center we adopt

$$\theta_a(0) = 1; \quad \theta_a'(0) = \frac{d\theta_a}{d\zeta}\mid_0 = 0; \quad a = 0, 1. \tag{31}$$

In tables 1 and 2, we summarize the numerical results for the Newtonian and post-Newtonian polytropes for different polytropic indices and q values. The pn corrections tend to reduce the radius of the polytrope. The higher the polytropic index the smaller this radius. The same is true, of course, for higher values of q.

8

5 Concluding remarks

As we discussed in section 1, some authors is debated to exist a new modes of oscillations in relativistic stellar systems. They believed that these modes generated by the perturbations in space time metric and no have analogue in Newtonian systems. They used the general relativistic hydrodynamics to distinguish them. Although this way is routine but one needs to assume some thermodynamic concepts that may be fault in relativistic regime. Hence, to reject these conceptual problems, we choosed general relativistic Liouville's equation that is the purely dynamical theory. The combination Liouville and Einstein equations enable one to study the behavior of relativistic systems without ambiguity. Therefore, in this paper, we used the *pn* approximation to obtain the Einstein-Liouville equation for a relativistic self gravitating stellar system. We found two integrals, generalization of the classical energy and angular momentum, that are satisfying *pnl* in equilibrium state. These solutions enable one to construct an equilibrium model for the system in *pn* approximation. Polytropic models, the most familiar stellar models, are constructed in *pn* approximation. In tables 1 and 2, we compared these models with its Newtonian correspondence. The *pn* corrections tend to reduce the radius of polytrope. The higher the polytropic index the smaller this radius. We introduced a parameter q, Eq. (29), to enter the effect of central density of system in calculations. Increasing values of q reduce the radius of system. In the second paper (Sobouti and Rezania 1998), we study time-dependent oscillations of a relativistic system in *pn* approximation.

Appendix A: Derivation of Eqs. (5)

Consider a general coordinate transformation (X, U) to (Y, V). The corresponding partial derivatives transform as

$$\begin{pmatrix} \partial/\partial X \\ \partial/\partial U \end{pmatrix} = M \begin{pmatrix} \partial/\partial Y \\ \partial/\partial V \end{pmatrix},$$

$$= \begin{pmatrix} \partial Y/\partial X & \partial V/\partial X \\ \partial Y/\partial U & \partial V/\partial U \end{pmatrix} \begin{pmatrix} \partial/\partial Y \\ \partial/\partial V \end{pmatrix}, \qquad \text{(A.1)}$$

where M is the 7×7 Jacobian matrix of transformation. Setting $X = Y = x^\mu$, $V = v^i$ and $U = U^i$ for our problem, one finds

$$M = \begin{pmatrix} \partial x^\mu/\partial x^\nu & \partial v^i/\partial x^\nu \\ \partial x^\mu/\partial U^j & \partial v^i/\partial U^j \end{pmatrix}, \qquad \text{(A.2a)}$$

and its inverse

$$M^{-1} = \begin{pmatrix} \partial x^\mu/\partial x^\nu & \partial U^i/\partial x^\nu \\ \partial x^\mu/\partial v^j & \partial U^i/\partial v^j \end{pmatrix}. \qquad \text{(A.2b)}$$

One easily finds

$$\partial x^\mu/\partial x^\nu = \delta_{\mu\nu}; \qquad \partial x^\mu/\partial v^j = 0, \qquad \text{(A.3a)}$$

$$\partial U^i/\partial x^\nu = v^i \partial U^0/\partial x^\nu = \frac{U^{0^3} v^i}{2} \frac{\partial g_{\alpha\beta}}{\partial x^\nu} v^\alpha v^\beta, \qquad \text{(A.3b)}$$

$$\partial U^i/\partial v^j = U^0 \delta_{ij} + v^i \partial U^0/\partial v^j = U^0 \delta_{ij} + U^{0^3} v^i g_{j\beta} v^\beta. \qquad \text{(A.3c)}$$

Inserting the latter in M^{-1} and inverting the result one arrives at M from which Eqs. (5) can be read out.

Appendix B: Post-Newtonian hydrodynamics

Mathematical manipulations in the development of this work has been tasking. To ensure that no error has crept in the course of calculations we have tried to infer the post-Newtonian hydrodynamical equations from the post-Newtonian Liouville equation derived earlier. From Eq. (6a) one has

$$
\begin{aligned}
\mathcal{L}_U^{pn} f &= U^0 (\mathcal{L}^{cl} + \mathcal{L}^{pn}) f \\
&= [(1 - \phi + \frac{1}{2} \mathbf{v}^2) \mathcal{L}^{cl} + \mathcal{L}^{pn}] f,
\end{aligned}
\tag{B.1}
$$

where $\mathcal{L}^{cl}$ and $\mathcal{L}^{pn}$ are given by Eq. (10). We integrate $\mathcal{L}_U^{pn} f$ over the $\mathbf{v}$-space:

$$
\int \mathcal{L}_U^{pn} f d^3 v = \int [(1 - \phi + \frac{1}{2} \mathbf{v}^2) \mathcal{L}^{cl} + \mathcal{L}^{pn}] f d^3 v.
\tag{B.2}
$$

Using Eqs. (12) and (13), one finds the continuity equation

$$
\frac{\partial}{\partial t} ({}^0 T^{00} + {}^2 T^{00}) + \frac{\partial}{\partial x^j} ({}^1 T^{0j} + {}^3 T^{0j}) - {}^0 T^{00} \frac{\partial \phi}{\partial t} = 0,
\tag{B.3}
$$

which is the *pn* expansion of the continuity equation

$$
T^{0\nu}{}_{;\nu} = 0,
\tag{B.4}
$$

Next, we multiply $\mathcal{L}_U^{pn} f$ by v^i and integrate over the $\mathbf{v}$-space:

$$
\int v^i \mathcal{L}_U^{pn} f d^3 v = \int v^i [(1 - \phi + \frac{1}{2} \mathbf{v}^2) \mathcal{L}^{cl} + \mathcal{L}^{pn}] f d^3 v.
\tag{B.5}
$$

After some calculations one finds

$$
\begin{aligned}
&\frac{\partial}{\partial t} ({}^1 T^{0i} + {}^3 T^{0i}) + \frac{\partial}{\partial x^j} ({}^2 T^{ij} + {}^4 T^{ij}) \\
&+ {}^0 T^{00} [\frac{\partial}{\partial x^i} (\phi + 2\phi^2 + \psi) + \frac{\partial \eta_i}{\partial t}] + {}^2 T^{00} \frac{\partial \phi}{\partial x^i} \\
&+ {}^1 T^{0j} (\frac{\partial \eta_i}{\partial x^j} - \frac{\partial \eta_j}{\partial x^i} - 4\delta_{ij} \frac{\partial \phi}{\partial t}) + {}^2 T^{jk} (\delta_{jk} \frac{\partial \phi}{\partial x^i} - 4\delta_{ik} \frac{\partial \phi}{\partial x^j}) = 0.
\end{aligned}
\tag{B.6}
$$

The latter is the correct *pn* expansion of

$$
T^{i\nu}{}_{;\nu} = 0; \quad i = 1, 2, 3.
\tag{B.7}
$$

See Weinberg 1972, QED.

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Table 1. A comparison of the Newtonian and post-Newtonian polytropes at certain selected radii for n=1, 2 and 3 and different values q.

Table 2. Same as Table 1. n=4 and 4.5.

Table 1.

n	Polytropic radius, ζ	Newtonian polytrope, θ	pn polytrope, $\theta + q(\Theta - 2\theta^2)$		
			$q = 10^{-5}$	$q = 10^{-3}$	$q = 10^{-1}$
1	0.0000000	1.00000	1.00000	1.00000	1.00000
	1.0000000	0.84145	0.84145	0.84143	0.83936
	2.0000000	0.45458	0.45458	0.45433	0.42949
	2.9233000	0.07408	0.07407	0.07334	0.00000
	3.1388500	0.00087	0.00086	0.00000	
	3.1415500	0.00001	0.00000		
	3.1415930	0.00000			
2	0.0000000	1.00000	1.00000	1.00000	1.00000
	1.0000000	0.84868	0.84868	0.84863	0.84394
	2.0000000	0.52989	0.52988	0.52945	0.48609
	3.0000000	0.24188	0.24187	0.24031	0.13289
	3.4737000	0.13904	0.13902	0.13770	0.00000
	4.3394800	0.00171	0.00169	0.00000	
	4.3527000	0.00002	0.00000		
	4.3529000	0.00000			
3	0.0000000	1.00000	1.00000	1.00000	1.00000
	1.0000000	0.85480	0.85480	0.85473	0.84773
	2.0000000	0.58284	0.58283	0.58230	0.52894
	3.0000000	0.35939	0.35938	0.35824	0.24016
	4.0000000	0.20927	0.20925	0.20764	0.03226
	4.1939500	0.18690	0.18688	0.18520	0.00000
	6.8435000	0.00228	0.00225	0.00000	
	6.8963000	0.00002	0.00000		
	6.8967000	0.00000			

Table 2.

n	Polytropic radius, ζ	Newtonian polytrope, θ	pn polytrope, $\theta + q(\Theta - 2\theta^2)$		
			$q = 10^{-5}$	$q = 10^{-3}$	$q = 10^{-1}$
4	0.0000000	1.00000	1.00000	1.00000	1.00000
	2.0000000	0.62294	0.62293	0.62235	0.56326
	4.0000000	0.31804	0.31802	0.31645	0.14194
	5.1541000	0.22574	0.22572	0.22383	0.00000
	8.0000000	0.10450	0.10448	0.10221	
	12.0000000	0.02972	0.02970	0.02716	
	14.0000000	0.00833	0.00830	0.00570	
	14.6468000	0.00265	0.00262	0.00000	
	14.9680000	0.00003	0.00000		
	14.9713400	0.00000			
4.5	0.0000000	1.00000	1.00000	1.00000	1.00000
	2.0000000	0.63965	0.63964	0.63905	0.57857
	4.0000000	0.36053	0.36651	0.35897	0.18628
	5.7468600	0.24334	0.24332	0.24135	0.00000
	8.0000000	0.16173	0.16171	0.15946	
	12.0000000	0.09015	0.09013	0.08766	
	16.0000000	0.05402	0.05399	0.05141	
	20.0000000	0.03230	0.03227	0.02962	
	24.0000000	0.01782	0.01779	0.01510	
	28.0000000	0.00747	0.00744	0.00472	
	30.2689200	0.00282	0.00279	0.00000	
	31.7792300	0.00004	0.00000		
	31.7878400	0.00000			