Angular momentum transfer to a star by gravitational waves

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Summary and Introduction

Interaction of a stochastic background of gravitational radiation with celestial systems changes their dynamical elements in a random manner and give rise to secular changes in time(Berttoti 1973, Mashhoon *etal* 1981, Khosroshahi and Sobouti 1997). It has been speculated that a study of such secular changes might serve as a possible mean of detecting gravitational radiation. In this spirit we study the angular momentum transfer from a random background of radiation either to a rotating star or to an oscillating one. The angular momentum transferred to such objects by a continuous plane wave is proportional to time, t, and by an stochastic background is proportional to $t^{1/2}$.

1. Interaction of a rotating star with gravitational waves

Consider a polytrope rotating with an angular frequency Ω, about the z-axis. The density ρ at position $\mathbf{r}$, may be written as

$$\rho(\mathbf{r}, b) = \rho_0(r) + b[\rho_1(r) + \rho_2(r)P_2(\cos\theta)] \tag{1}$$

where $b = \Omega^2/\omega_{osc}^2$ is an expansion parameter and $\omega_{osc} = \sqrt{4\pi G\rho(0)}$ is of the order of the angular frequency of natural oscillations of the star. Suppose that a monochromatic plane gravitational wave with frequency ω falls on the star in a direction characterized by azimutal and polar angles α and β, respectively. Assume the radius of the star is much smaller than the wavelength of the incident wave. In a Fermi coordinate system, the tidal force on unit mass exerted by the wave is the gradient of a scalar,

$$\mathbf{f}^{gw} = -\nabla\Phi = \nabla[\frac{1}{4}\omega^2\epsilon\sum_{i,j=1}^{3} x_i h_{ij}(t) x_j], \tag{2}$$

where $\epsilon \ll 1$ is a certain characteristic amplitude of the wave and is independent of the frequency ω and $h_{ij}(t) = Re\{A_{ij}(\omega)exp(-i\omega t + i\mathbf{k}.\mathbf{x})\}$ are the spatial components of fluctuations in the space-time metric by the incident wave. h_{ij} can be taken to be transverse and traceless. The torque exerted on the star, measured by a corotating observer, is

$$\vec{\tau} = \frac{d\mathbf{L}}{dt} = \int_v \mathbf{r}\times\rho\mathbf{f}^{gw}d^3\mathbf{r} = \frac{\pi}{30}\frac{MR^2}{\bar{\rho}}b\epsilon A(\omega)\omega^2\int_0^1 \rho_2(x)x^4dx\{h_1(t)\hat{i} + h_2(t)\hat{j}\} \tag{3a}$$

where $\bar{\rho}, M, R$ are the mean density, mass and radius of the star, respectively,

$$h_1(t) = \frac{1}{2}\sin 2\beta\sin(\Omega t + \alpha)\cos\omega t - \sigma\sin\beta\cos(\Omega t + \alpha)\sin\omega t, \tag{3b}$$

$$h_2(t) = \frac{1}{2}\sin 2\beta\cos(\Omega t + \alpha)\cos\omega t + \sigma\sin\beta\sin(\Omega t + \alpha)\sin\omega t, \tag{3c}$$

and $\sigma = 1$, and -1 for right and left circular polarization, respectively. We substitute eqs.(3b,c) in (3a) and integrate with respect to time. In resonance the secular change in the angular momentum is proportional to t. Furthermore, this change is in the xy-plane and manifests itself as a precession of the rotation axis. In an isotropic background of radiation, the mean precession amplitude is zero. Its root mean square value, however, changes as $t^{1/2}$. Thus,

$$\theta^2_{rms} \simeq \frac{L^2_{xy}}{L^2_z} \simeq \frac{1}{L^2_z}\int d(\cos\beta)d\alpha[\int_0^t \vec{\tau}dt]^2 d\omega, \tag{4a}$$

and finally,

$$\theta^2_{rms} \sim \frac{\pi G b^2}{\bar{\rho}}[\int_0^1 \rho_{b2}(x)x^4dx]^2\mathcal{S}_g(\Omega)t \qquad\qquad for \;\; t \gg \Omega^{-1}, \tag{4b}$$

where $\mathcal{S}_g(\omega) = \epsilon^2 A(\omega)^2\omega^2/8\pi G$ is the energy spectral density of the background radiation. Equation(4) indicates a random walk for the rotation axis of the star.

2. Interaction with normal modes of a star

The torque of a gravitational wave on a rotating star, eqs.(3), is due to deviation from the spherical symmetry of the star. This asymmetry may come about because of the natural oscillation of the star. Helioseismic waves in different spherical harmonic numbers are prominent examples of such occurrence. This is the justification for the following analysis .

Let $\rho(r)$, $p(r)$ and $U(r)$ denote the density, the pressure and the gravitational potential of a star in hydrostatic equilibrium. Let a mass element at $\mathbf{r}$ undergo an infinitesimal displacement $\xi(\mathbf{r},t)$ from its equilibrium position. It causes small changes $\delta\rho(\mathbf{r},t), \delta p(\mathbf{r},t)$ and $\delta U(\mathbf{r},t)$. The linearized Euler's equation of motion is

$$-\rho\ddot{\xi} = \nabla(\delta p) + \delta\rho\nabla\Omega + \rho\nabla(\delta U) = \mathcal{W}\xi \;, \tag{5}$$

where,

$$\delta\rho = -\nabla.(\rho\xi), \tag{6a}$$

$$\delta p = \frac{dp}{d\rho}\delta\rho - [(\frac{\partial p}{\partial\rho})_{ad} - \frac{dp}{d\rho}]\rho\nabla.\xi, \tag{6b}$$

$$\nabla^2(\delta U) = -4\pi G\delta\rho. \tag{6c}$$

The displacement ξ belongs to a function space $\mathcal{H}$ in which the inner product is defined as $(\eta, \rho\xi) = \int \rho\eta^*.\xi\ d^3x = finite, \quad \xi, \eta \in \mathcal{H}$. The operator $\mathcal{W}$ is self-adjoint on $\mathcal{H}$ and gives rise to the eigenvalue problem $\mathcal{W}\xi_n = \omega_n^2\rho\xi_n$, where ω_n^2 are real. Using a gauged version of Helmholtz's theorem, one may decompose a general vector into an irrotational and a "weighted" solenoidal component. Thus

$$\xi = \xi_p + \xi_g, \tag{7a}$$

where

$$\xi_p = -\nabla\chi_p; \qquad with \ \ \nabla \times \xi_p = 0, \tag{7b}$$

$$\xi_g = \rho^{-1}\nabla \times \nabla \times (\hat{\mathbf{r}}\chi_g); \qquad with \ \ \nabla.(\rho\xi_g) = 0. \tag{7c}$$

Here $\hat{\mathbf{r}}$ is the unit vector in $\mathbf{r}$ direction, and χ_p and χ_g are two scalars. See Sobouti(1980) for details of eqs.(5-7). As in eqs.(3) the torque exerted by a gravitational wave incident on a star is,

$$\frac{d\mathbf{L}}{dt} = \int_v \delta\rho\mathbf{r} \times \mathbf{f}^{gw} d^3x. \tag{8}$$

From eqs.(6a) and (7), $\delta\rho$ for ξ_g is zero and for ξ_p is

$$\delta\rho = \rho(r)\{\chi_p'' + (\frac{2}{r} + \frac{\rho'}{\rho})\chi_p' - \frac{l(l+1)}{r^2}\chi_p\}Y_{lm}(\theta,\phi)e^{-i\omega_n t}. \tag{9}$$

Only $l = 2$ modes will contribute to the torque and for simplicity we will consider the $m = 0$ case. Equation(8) reduces to an expression similar to that of eqs.(3),

$$\frac{d\mathbf{L}}{dt} = -\frac{1}{9}\frac{MR^2}{\bar{\rho}}A(\omega)\omega^2\int_0^1 \rho(x)\{\chi_p'' + (\frac{2}{x} + \frac{\rho'}{\rho})\chi_p' - \frac{6}{x^2}\chi_p\}x^4 dx \times \{h_1'(t)\hat{i} + h_2'(t)\hat{j}\}\cos\omega_n t, \tag{10a}$$

where

$$h_1'(t) = -\frac{1}{2}\sin 2\beta \sin\alpha\cos\omega t + \sigma\sin\beta\cos\alpha\sin\omega t \tag{10b}$$

$$h_2'(t) = -\frac{1}{2}\sin 2\beta \sin\alpha \cos\omega t - \sigma \sin\beta \cos\alpha \sin\omega t \tag{10c}$$

The root mean square of the amplitude of precession induced by an isotropic background of radiation becomes

$$\theta^2_{rms} \sim \frac{\pi}{\bar{\rho}}[\int_0^1 \rho(x)\{\chi'' + (\frac{2}{x} + \frac{\rho'}{\rho})\chi' - \frac{6}{x^2}\chi\}x^4 dx]^2 \mathcal{S}_g(\omega_n)t \tag{11}$$

$$for \quad t \gg \omega_n^{-1}$$

A numerical evaluation of eq.(11) involves the following steps.

a) One calculates the p and g modes of the model star belonging to $l = 2$. A $p-$mode will have a large irrotational component as indicated in eqs(1). For a $g-$mode the situation will be the opposite.

b) One extracts the irrotational component, χ_p of each mode, substitutes in eq.(11) and carries out the integration numerically. Numerical evaluations of the rms precession amplitudes are in progress.

References

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