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A LIE ALGEBRA OF THE SYMMETRIES OF LIOUVILLE'S EQUATION

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INTRODUCTION

It is an assumption of traditional stellar dynamics that Liouville's equation governs the time evolution of stellar systems. An inevitable consequence of such a premise is that (a) at least some modes of instability of stellar systems may be those of Liouville's equation; and (b) stellar systems might undergo periodic changes of definite patterns in configuration and velocity spaces. For, Liouville's equation exhibits eigenmodes of oscillation. While it is not feasible to observe the astronomically long periods of oscillations, the patterns of changes, i.e. the eigenfunctions, may be amenable to observation by analyzing the COD records of brightness and velocity distributions on the visible disks of galaxies and globular clusters. This is our motivation for scrutinizing Liouville's equation, if not for its own merits. The work is a continuation of a series of papers on the symmetries of Liouville's equation (Sobouti 1989a,b, Sobouti and Samimi 1989).

SYMMETRY TRANSFORMATION

Liouville's equation may be written as follows:

$$Lf = i\frac{\partial f}{\partial t}\ ;\ L = i\left(p_i\frac{\partial}{\partial q_i} - \frac{\partial U}{\partial q_i}\frac{\partial}{\partial p_i}\right), \tag{1}$$

where (q,p) is the collection of configuration and momentum coordinates, respectively, U(q) is the potential, L is Liouville's operator, and f(q,p,t) is in general a complex valued function of (q,p,t). The latter is a member of a function

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space, a Hilbert space H, in which the inner product is defined as $(f,g)=\int f^*g\,dq\,dp$; $f,g \in H$. The reason for introducing the imaginary number i is to render L hermitian in the sense $(Lf,g) = (f,Lg)$; $f,g \in H$. For a systematic study of the symmetry of Eq.(1) we follow a procedure parallel to that of Killing to obtain the isometries of curved space times. We look for those infinitesimal transformations of (q,p) to (q',p') that leave L form invariant, hence let

$$q_i = q'_i - \epsilon\xi_i(q',p'), \tag{2a}$$

$$p_i = p'_i - \epsilon\eta_i(q',p'), \tag{2b}$$

where ϵ is an infinitesimal parameter. The symmetry requirement is

$$L'(q',p') = L(q',p'), \tag{3}$$

where $L'(q',p')$ is the transformed Liouville's operator. By straightforward calculation Eq.(3) leads to the following differential equation for ξ_i and η_i:

$$L\xi_i + i\eta_i = 0, \tag{4a}$$

$$L\eta_i - i\xi_j \frac{\partial^2 U}{\partial q_j \partial q_i} = 0. \tag{4b}$$

Upon elimination of η, one obtains

$$L^2\xi_i - \xi_j \frac{\partial^2 U}{\partial q_j \partial q_i} = 0. \tag{4c}$$

It is worthwhile to note that all transformations generated by solutions of Eqs.(4) are canonical, in the sense that they leave the Poisson brackets $i[H,f] = Lf$ form invariant. As a consequence of the infinitesimal transformations, a phase space function changes as $f(q',p') = f(q,p) + \epsilon\chi f(q,p)$; $f \in H$, where the generator χ is given by

$$\chi = \xi_i \frac{\partial}{\partial q_i} + \eta_i \frac{\partial}{\partial p_i} \tag{5}$$

One may readily verify that

$$|L,\chi| = 0. \tag{6}$$

So much for generalities. Further progress requires specific assumptions with regard the potential U{q}. Sobouti (1989a,b)

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has shown that for spherically symmetric potentials there are three generators that obey the angular momentum algebra and concludes that the symmetry group of L is So(3). He also gives some symmetry operators for quadratic potentials, but not all. Here, we study the latter case in detail and provide the full symmetry group.

MAXIMALLY SYMMETRIC QUADRATIC POTENTIALS

The case to be studied, as any other aspect of quadratic potentials is exactly and analytically solvable. It unravels many of the idiosyncrasies of questions and answers that arise in the course of analysis. Apart from their academic merits, however, quadratic potentials do find important applications. The central regions of extended stellar systems, the de Sitter and anti de Sitter spacetimes are examples. Furthermore, the complete eigensolutions of Liouville's equation for quadratic potentials can be used either as a basis for the function space H or as approximate solutions for less symmetric potentials.

For $U = \frac{1}{2} q_i q_j$, solutions of Eqs.(4) are

$$\xi_i = a_{ij} q_j + b_{ij} p_j, \tag{7a}$$

$$\eta_i = -b_{ij} q_j + a_{ij} p_j, \tag{7b}$$

where a_{ij} and b_{ij} are eighteen real constants. The generators of Eq.(5) become

$$X_\alpha = -i\{ a_{jk}(q_j \frac{\partial}{\partial q_k} + p_j \frac{\partial}{\partial p_k}) + b_{jk}(p_j \frac{\partial}{\partial q_k} - q_j \frac{\partial}{\partial p_k})\}, \tag{8}$$

where $\alpha = 1,2,\dots,18$, correspond to eighteen independent choices of a_{ij} and b_{ij}. Straightforward calculations show that X_α's obey a Lie algebra. That is, one finds

$$|X_\alpha, X_\beta| = C^\gamma_{\alpha\beta} X_\gamma \; ; \quad \alpha,\beta,\gamma = 1,2,\dots,18 \tag{9a}$$

where $C^\gamma_{\alpha\beta}$'s are real structure constants and, as a consequence of Jacobi identity for commutator brackets satisfy the following (see e.g. Wybourne, 1974).

$$C^\delta_{\alpha\beta} C^\rho_{\gamma\delta} + C^\delta_{\beta\gamma} C^\rho_{\alpha\delta} + C^\delta_{\gamma\alpha} C^\rho_{\beta\delta} = 0. \tag{9b}$$

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This eighteen parameter Lie group is GL(3,C) which is neither simple nor semisimple. There are certain notable subgroups to it, discussed in the next section.

CLASSIFICATION OF THE SYMMETRIES

For each 3×3 matrix a_{ij} and b_{ij} we choose one unit matrix, three anisymmetric matrices and five traceless symmetric matrices. For $a_{ij} = 0$ and $b_{ij} = \delta_{ij}$ the generator χ turns out to be Liouville's operator itself. The following subalgebras are identified.

Rotational invariance group, SO(3)

The transformations for which $a_{jk} = -a_{jk} - \varepsilon_{ijk}$, levi civita symbol, and $b_{ij} = 0$ are rotations in p-p and q-q planes. They leave q^2 and p^2 invariant. The three generators are

$$J_i = - i\varepsilon_{ijk}(p_j \frac{\partial}{\partial p_k} + q_j \frac{\partial}{\partial q_k}), \tag{10a}$$

with the angular momentum algebra

$$[J_i, J_j] = i\varepsilon_{ijk} J_k \tag{10b}$$

The corresponding Cassimir operator, that is the operator which commutes with all J_i's is J^2. Sobouti (1989a,b) has shown that this SO(3) symmetry holds true for any spherically symmetric potential. Sobouti and Samimi (1989) have demonstrated the same for the linearized Liouville's equation of spherically smmetric stellar systems and utilized in the computations of normal modes of oscillations.

Invariance group of Lagrangian SO(3,1)

The transformations with $b_{jk} = \varepsilon_{ijk}$ are Lorentz boosts in p-q planes. Together with $a_{jk} = \varepsilon_{ijk}$, they leave the Lagrangian, $\frac{1}{2}(p^2-q^2)$, form invariant. The corresponding generators in additi J_i's are

$$I_i = - i\varepsilon_{ijk}(p_j \frac{\partial}{\partial q_k} - q_j \frac{\partial}{\partial p_k}); \text{ for } b_{jk} = \varepsilon_{ijk} \tag{11a}$$

One readily verifies that the act (J_1, I_1) has the closed Lie algebra of SO(2,1). That is in addition to Eq.(10b).

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$$[J_i, I_j] = i\varepsilon_{ijk} I_k \, , \tag{11b}$$

$$[I_i, I_j] = -i\varepsilon_{ijk} J_k \, . \tag{11c}$$

The two Casimir operators of this subalgebra are $J^2 - I^2$ and $J.I$.

Invariance group of Hamiltonian, SU(3)

This group has been considered by many authors and mainly for quantum mechanical and field theoretic purposes. Notables are the investigation of Jauch and Hill (1940), Fradkin (1964), Barut (1965) and Hwa and Nuyts (1966).

Invariance of the Hamiltonian, $H = \frac{1}{2}(p^2 + q^2)$, is realized for antisymmetric $a_{jk} = \varepsilon_{ijk}$ and symmetric b_{jk}'s. The former is already discussed. The latter can be chosen as a unit matrix and five traceless symmetric matrices. The corresponding operators are

$$L = -i\left(p_i \frac{\partial}{\partial q_i} - q_i \frac{\partial}{\partial p_i}\right) \; ; \quad \text{for } b_{jk} = \delta_{jk} \tag{12a}$$

$$M_i = -i|\varepsilon_{ijk}|\left(p_j \frac{\partial}{\partial q_k} - q_j \frac{\partial}{\partial p_k}\right); \text{ for } b_{jk} = |\varepsilon_{ijk}| \tag{12b}$$

$$P = -i\left(p_1 \frac{\partial}{\partial q_1} - p_2 \frac{\partial}{\partial q_2} - q_1 \frac{\partial}{\partial p_1} + q_2 \frac{\partial}{\partial p_2}\right);$$
$$\text{for } b_{jk} = \text{diagonal} = (1,-1,0), \tag{12c}$$

$$Q = -\frac{i}{\sqrt{3}}\left(p_1 \frac{\partial}{\partial q_1} + p_2 \frac{\partial}{\partial q_2} - 2p_3 \frac{\partial}{\partial q_3} - q_1 \frac{\partial}{\partial p_1} - q_2 \frac{\partial}{\partial p_2}\right.$$
$$\left. + 2q_3 \frac{\partial}{\partial p_3}\right); \text{ for } b_{jk} = \text{diagonal} = (1,1,-2). \tag{12d}$$

The set $\{J_i, M_i, P, Q\}$ has the Lie algebra of SU(3). It is worthwhile to mention that all of these nine transformations are orthogonal and can be understood as rotation in real phase space. The quadratic Casimir operator of this subalgebra is $J^2 + M^2 + P^2 + Q^2$.

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A further subalgebra, SU(2,1)

All the above sections dealt with symmetries induced by antisymmetric a_{ij}'s and symmetric and antisymmetric b_{ij}'s. There remains the symmetries belonging to symmetric a_{ij}'s. Again we choose the symmetric matrix a_{ij} as a unit matrix and five traceless symmetric matrices. The corresponding generators are

$$K = -i(p_i \frac{\partial}{\partial p_i} + q_i \frac{\partial}{\partial q_i}) \qquad ; \text{ for } a_{ij} = \delta_{ij}, b_{ij} = 0, \tag{13a}$$

$$N_i = -i|\varepsilon_{ijk}|(p_j \frac{\partial}{\partial p_k} + q_j \frac{\partial}{\partial q_k}); \qquad \text{for } a_{jk} = |\varepsilon_{ijk}| \tag{13b}$$

$$R = -i(p_1 \frac{\partial}{\partial p_1} - p_2 \frac{\partial}{\partial p_2} + q_1 \frac{\partial}{\partial q_1} - q_2 \frac{\partial}{\partial q_2});$$
$$\text{for } a_{jk} = \text{diagonal} = (1,-1,0), \tag{13c}$$

$$S = -\frac{i}{\sqrt{3}}(p_1 \frac{\partial}{\partial p_1} + p_2 \frac{\partial}{\partial p_2} - 2p_3 \frac{\partial}{\partial p_3} + q_1 \frac{\partial}{\partial q_1} + q_2 \frac{\partial}{\partial q_2}$$
$$- 2q_3 \frac{\partial}{\partial q_3}); \text{ for } a_{jk} = \text{diagonal} = (1,1,-2). \tag{13d}$$

The algebra of the eight generators $\{J_i, N_i, R, S\}$ is SU(2,1). The quadratic Casimir operator of this subalgebra is $J^2 - N^2 + S^2 - R^2$. The generator K corresponds to a scale transformation, and commutes with all the other operators. Therefore the algebra of the set $\{K, J_i, N_i, R, S\}$ is not simple. Furthermore, K is not hermitian but $K' = K-3i$ ($i = \sqrt{-1}$) is hermitian, $(g, K'f) = (K'g, f)$; $f, g \in H$. The generators N_i corresponds to Lorentz boosts in p-p and q-q planes.

A COMPLETE SET OF MUTUALLY COMMUTING OPERATORS

The eigenvalue equation, $Lf_\omega = \omega f_\omega$, has highly degenerate eigenvalues. For, in view of the constants of motion(energy, angular momentum, say), any f_ω multiplied by an arbitrary function of these constants is again an eigenfunction belonging to the same ω. A complete set of mutually commuting operators, will be found useful in classifying these degenerate sets. For the maximally symmetric case of the six dimensional phase space there should exists six such operators,

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including L itself. Among the eighteen generators and the Casimir operators introduced so far the set $\{L, J^2 - I^2, J, I, J_3, I_3, K'\}$ are hermitian, are linearly independent, and commute two by two. This set, however, is not unique. Khalesi (1990), for example, has proposed his own set and has obtained a complete set of nondegenerate eigensolutions of L. The incomplete set $\{L, J^2, J_3\}$ has also been utilized by Sobouti and Samimi.

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