Computational Data Mining

Part 3: Linear Algebra

Linear Systems and Least Squares

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Introduction to linear systems

we briefly review some facts about the solution of linear systems of equations,

$$
\begin{aligned}
a_{11}\boldsymbol{x_1} + a_{12}\boldsymbol{x_2} + a_{13}\boldsymbol{x_3} &= \boldsymbol{y_1} \\
a_{21}\boldsymbol{x_1} + a_{22}\boldsymbol{x_2} + a_{23}\boldsymbol{x_3} &= \boldsymbol{y_2} \\
a_{31}\boldsymbol{x_1} + a_{32}\boldsymbol{x_2} + a_{33}\boldsymbol{x_3} &= \boldsymbol{y_3}
\end{aligned}
$$

$$
\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}
\begin{bmatrix} \boldsymbol{x_1} \\ \boldsymbol{x_2} \\ \boldsymbol{x_3} \end{bmatrix}
=
\begin{bmatrix} \boldsymbol{y_1} \\ \boldsymbol{y_2} \\ \boldsymbol{y_3} \end{bmatrix}
$$

$$\mathrm{Ax} = \mathrm{y}$$

where $A \in R^{n\times n}$ is square and nonsingular.

Introduction to linear systems

Definition (**Triangular Matrices**)

An $n \times n$ matrix is said to be upper triangular if $a_{ij} = 0$ for $i > j$ and lower triangular if $a_{ij} = 0$ for $i < j$. Also A is said to be triangular if it is either upper triangular or lower triangular.

Example: $\begin{bmatrix} 1 & 0 & 0 \\ 7 & 3 & 0 \\ 2 & 4 & 3 \end{bmatrix}$ $\begin{bmatrix} 1 & 2 & 7 \\ 0 & 3 & 4 \\ 0 & 0 & 3 \end{bmatrix}$

Definition (**Diagonal Matrices**)

An $n \times n$ matrix is diagonal if $a_{ij} = 0$ whenever $i \neq j$.

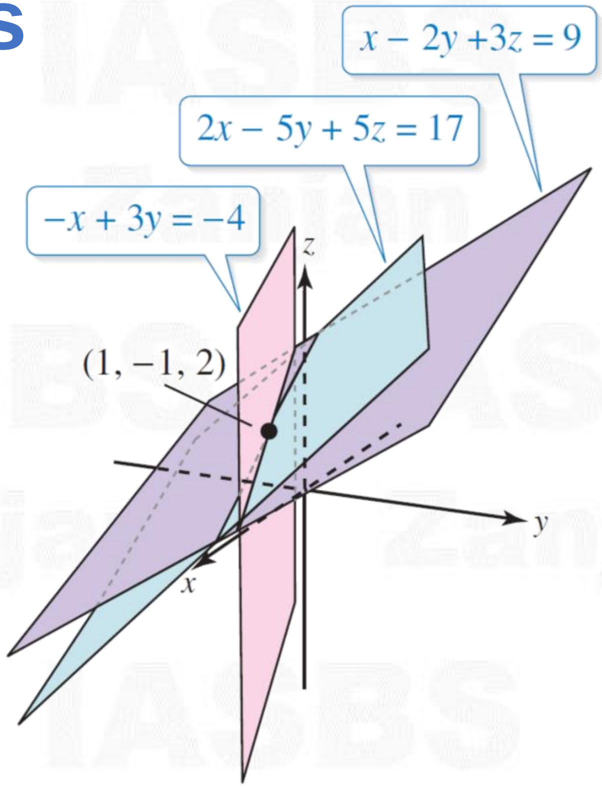
Example: $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

Introduction to linear systems

$$\begin{aligned} x - 2y + 3z &= 9 \\ -x + 3y &= -4 \\ 2x - 5y + 5z &= 17 \end{aligned}$$

$$\begin{bmatrix} 1 & -2 & 3 & 9 \\ -1 & 3 & 0 & -4 \\ 2 & -5 & 5 & 17 \end{bmatrix}$$

Augmented matrix



These kind of linear systems can be solved using Gaussian elimination

Operations That Produce Equivalent Systems

Each of these operations on a system of linear equations produces an *equivalent* system.

1. Interchange two equations.
2. Multiply an equation by a nonzero constant.
3. Add a multiple of an equation to another equation.

Introduction to linear systems

Gaussian elimination

$$\begin{bmatrix} 1 & -2 & 3 & 9 \\ -1 & 3 & 0 & -4 \\ 2 & -5 & 5 & 17 \end{bmatrix}$$

Adding the 1st row to the 2nd produces a new 2nd row.

$$\begin{bmatrix} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 2 & -5 & 5 & 17 \end{bmatrix}$$

Adding −2 times the 1st row to the 3rd row produces a new 3rd row

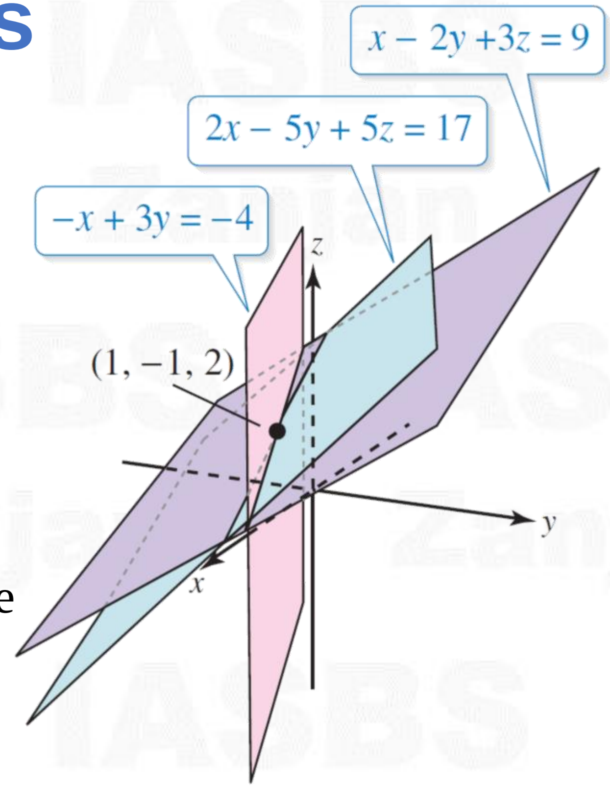
$$\begin{bmatrix} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 0 & -1 & -1 & -1 \end{bmatrix}$$

Adding the 2nd equation to the 3rd row produces a new 3rd row.

$$\begin{bmatrix} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 2 & 4 \end{bmatrix}$$

Multiplying the 3rd row by ½ produces a new 3rd row.

$$\begin{bmatrix} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$



$$\begin{aligned} x - 2y + 3z &= 9 \\ -x + 3y &= -4 \\ 2x - 5y + 5z &= 17 \end{aligned}$$

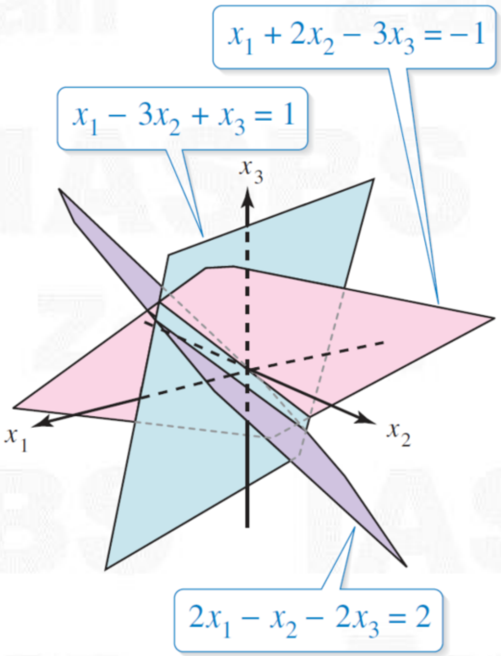
Gaussian Elimination with Partial Pivoting

Introduction to linear systems

A system of equations is considered overdetermined if there are more equations than unknowns:
The matrix $A \in R^{m \times n}$ is rectangular with $m > n$

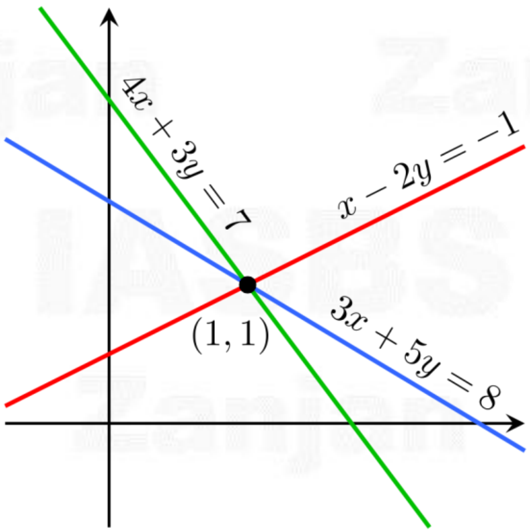
- A linear system is **inconsistent** if it has no solution otherwise it is said to be **consistent**

- The equations of a linear system are **independent** if none of the equations can be derived algebraically from the others.

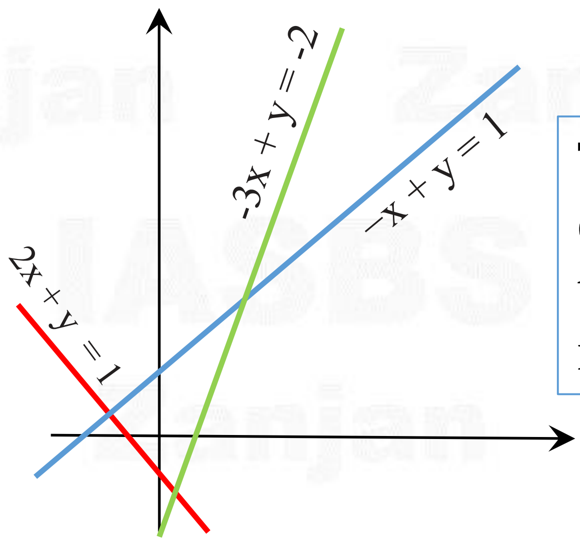


Introduction to linear systems

x − 2y = -1
3x + 5y = 8
4x + 3y = 7



− x + y = 1
2x + y = 1
-3x + y = -2



$$\begin{bmatrix} 1 & -2 & -1 \\ 3 & 5 & 8 \\ 4 & 3 & 7 \end{bmatrix} \quad \text{Rank} = 2$$

$$\begin{bmatrix} -1 & 1 & 1 \\ 2 & 1 & 1 \\ -3 & 1 & -2 \end{bmatrix} \quad \text{Rank} = 3$$

The number of independent equations in a system equals the rank of the augmented matrix of the system

If a system has more independent equations than unknowns, it is inconsistent and has no solutions.

Introduction to linear systems

Let $A \in R^{n\times n}$ and assume that A is nonsingular. Then for any right-hand-side b, the linear system $Ax = b$ has a unique solution.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} \boldsymbol{x_1} \\ \boldsymbol{x_2} \\ \boldsymbol{x_3} \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

A square matrix $A \in R^{n\times n}$ with rank n is called nonsingular and has an inverse A^{-1} satisfying.

LU Decomposition

The observation of linear systems ($Ax=b$) involving triangular coefficient matrices are easier to deal with.

Definition (**LU factorization**)

- If the $n \times n$ matrix A can be written as the product of a lower triangular matrix L and an upper triangular matrix U, then we can write:

$$A = LU$$

where L is a lower triangular matrix and U is an upper triangular matrix.

LU Decomposition

Definition (**an Elementary Matrix**)
An $n \times n$ matrix is an elementary matrix when it can be obtained from the identity matrix I_n by a single elementary row operation.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 \\ 0 & ½ & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_2 \leftrightarrow R_3$ $\qquad$ $(-2)R_1 + R_3 \rightarrow R_3$ $\qquad$ $½R_2 \rightarrow R_2$

Definition (**Row Equivalence**)
Let **A** and **B** be $m \times n$ matrices. Matrix **B** is **row-equivalent** to **A** when there exists a finite number of elementary matrices $\mathbf{E}_1, \mathbf{E}_2, \ldots, \mathbf{E}_k$ such that $\mathbf{B} = \mathbf{E}_k\mathbf{E}_{k-1} \ldots \mathbf{E}_2\mathbf{E}_1\mathbf{A}$.

LU Decomposition

$$A = LU$$

Begin by row reducing **A** to upper triangular form while keeping track of the elementary matrices used for each row operation.

Example:

$$\mathbf{A} \underset{\mathbf{E}_1}{\rightarrow} \underset{\mathbf{E}_2}{\rightarrow} \mathbf{U}$$

$$\mathbf{E}_2\mathbf{E}_1\mathbf{A} = \mathbf{U}$$

$$\mathbf{E}_2^{-1}\mathbf{E}_2\mathbf{E}_1\mathbf{A} = \mathbf{E}_2^{-1}\mathbf{U}$$

$$\mathbf{E}_1^{-1}\mathbf{E}_1\mathbf{A} = \mathbf{E}_1^{-1}\mathbf{E}_2^{-1}\mathbf{U}$$

$$\mathbf{A} = \mathbf{E}_1^{-1}\mathbf{E}_2^{-1}\mathbf{U}$$

$$\mathbf{A} = \mathbf{L}\mathbf{U}$$

L is the product of the inverses of the elementary matrices used in the row reduction.

LU Decomposition

Example:

$$\mathbf{A} = \begin{bmatrix} 1 & -3 & 0 \\ 0 & 1 & 3 \\ 2 & -10 & 2 \end{bmatrix} \quad E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \quad E_1^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$$

$R_3 + (-2)R_1 \rightarrow R_3$

$$\begin{bmatrix} 1 & -3 & 0 \\ 0 & 1 & 3 \\ 0 & -4 & 2 \end{bmatrix} \quad E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 4 & 1 \end{bmatrix} \quad E_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -4 & 1 \end{bmatrix}$$

$R_3 + (4)R_2 \rightarrow R_3$

$$\begin{bmatrix} 1 & -3 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 14 \end{bmatrix} = \mathbf{U} \qquad E_1^{-1}E_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & -4 & 1 \end{bmatrix} = \mathbf{L}$$

LU Decomposition

Once an LU-factorization of a matrix A was obtained, the system of n linear equations in n variables **Ax = b** can be solve very efficiently in two steps:

1. let **y = Ux** and solve **Ly = b** for **y**.
2. Solve **Ux = y** for x.

Example:

$$x_1 - 3x_2 = -5$$
$$x_2 + 3x_3 = -1$$
$$2x_1 - 10x_2 + 2x_3 = -20$$

$$\mathbf{A} = \begin{bmatrix} 1 & -3 & 0 \\ 0 & 1 & 3 \\ 2 & -10 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & -4 & 1 \end{bmatrix} \begin{bmatrix} 1 & -3 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 14 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -3 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 14 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} -5 \\ -1 \\ -20 \end{bmatrix} \qquad \mathbf{y} = \begin{bmatrix} -5 \\ -1 \\ -14 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & -4 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -5 \\ -1 \\ -14 \end{bmatrix}$$

$$\mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

LU Decomposition

Any Question?