Computational Data Mining

Part 4: Linear Algebra

Matrix Decomposition

Instructor: Zahra Narimani

Department of Computer Science and Information Technology,
Institute for Advanced Studies in Basic Sciences, Zanjan, Iran

Positive Matrix

- A matrix in which all the elements are greater than zero

Positive-definite matrix

- A symmetric nxn matrix is said to be positive definitive if the scalar $\mathbf{z}^T\mathbf{M}\mathbf{z}$ is strictly positive for **every non-zero column vector z** of n real numbers.

LDL^T decomposition

- Any symmetric, positive definite matrix A has a decomposition

$$A = LDL^T$$

- L: lower triangular with ones on the main diagonal
- D: diagonal matrix with positive diagonal elements

Example

$$A = \begin{pmatrix} 8 & 4 & 2 \\ 4 & 6 & 0 \\ 2 & 0 & 3 \end{pmatrix}$$

$$A = LU = \begin{pmatrix} 1 & 0 & 0 \\ 0.5 & 1 & 0 \\ 0.25 & -0.25 & 1 \end{pmatrix} \begin{pmatrix} 8 & 4 & 2 \\ 0 & 4 & -1 \\ 0 & 0 & 2.25 \end{pmatrix}$$

LDL^T Decomposition of A

$$A = LDL^T, \quad D = \begin{pmatrix} 8 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 2.25 \end{pmatrix}$$

$$A = LDL^T = (LD^{1/2})(D^{1/2}L^T) = U^TU$$

- The diagonal elements in D are positive

$$D^{1/2} = \begin{pmatrix} \sqrt{d_1} & & & \\ & \sqrt{d_2} & & \\ & & \ddots & \\ & & & \sqrt{d_n} \end{pmatrix}$$

- This variant of the LDLT decomposition is called the Cholesky decomposition.

Floating Point Computations

- The execution times of different algorithms
  - Can be compared by counting the number of floating point operations
- For scalars y=y+a*x
  - two flops

Cholesky decomposition

- **A is symmetric:** store only the main diagonal and elements above it n(n+1)/2
- Only **half** as many elements as in the **ordinary LU** decomposition need to be computed
- The amount of work is also halved—approximately **$n^3/3$** flops

Floating Point Rounding Errors

- A real number x, in general, cannot be represented exactly in a floating point system.

$$fl[x] = x(1 + \varepsilon)$$

$$\left| \frac{fl[x] - x}{x} \right| \leq \mu$$

- any real number (floating point system) is represented with a relative error not exceeding the **unit round-off μ**

$$fl[x \odot y] = (x \odot y)(1+\varepsilon)$$

$$fl[x \odot y] = (x + e) \odot (y + f)$$

$fl[x \odot y]$ is the exact result of the operation on slightly perturbed data!

Rounding Errors in Gaussian Elimination

- Small numbers on the main diagonal

$$\begin{bmatrix} 0.00001 & 0.2 \\ 0 & 5999.6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0.2 \\ 5999.9 \end{bmatrix} \Rightarrow \boxed{\begin{matrix} x_2 = 1.00000500 \\ x_1 = -1.00006667 \end{matrix}}$$

- A * $[x_1, x_2]$ = [0.199991, 5999.629998]

Ill-Conditioned

- A mathematical problem or series of equations is ill-conditioned if a small change in the independent variable (input) leads to a large change in the dependent variable (output)

$$\begin{cases} x_1 + 2x_2 = 10 \\ 1.1x_1 + 2x_2 = 10.4 \end{cases}$$

$$x_1 = \frac{2(10) - 2(10.4)}{1(2) - 2(1.1)} = 4 \qquad x_2 = \frac{1(10.4) - 1.1(10)}{1(2) - 2(1.1)} = 3$$

a_{21} from 1.1 to 1.05

$$x_1 = \frac{2(10) - 2(10.4)}{1(2) - 2(1.05)} = 8 \qquad x_2 = \frac{1(10.4) - 1.1(10)}{1(2) - 2(1.05)} = 1$$

Singular system

- Having two or more equal equations
- There exists a zero element in main diagonal after elementary row operations (even with pivoting)

Solutions

- Use double precision
- Pivoting
- Scaling

Any Question?