Computational Data Mining

Part 1: Linear Algebra

Vector Spaces

Change of Basis

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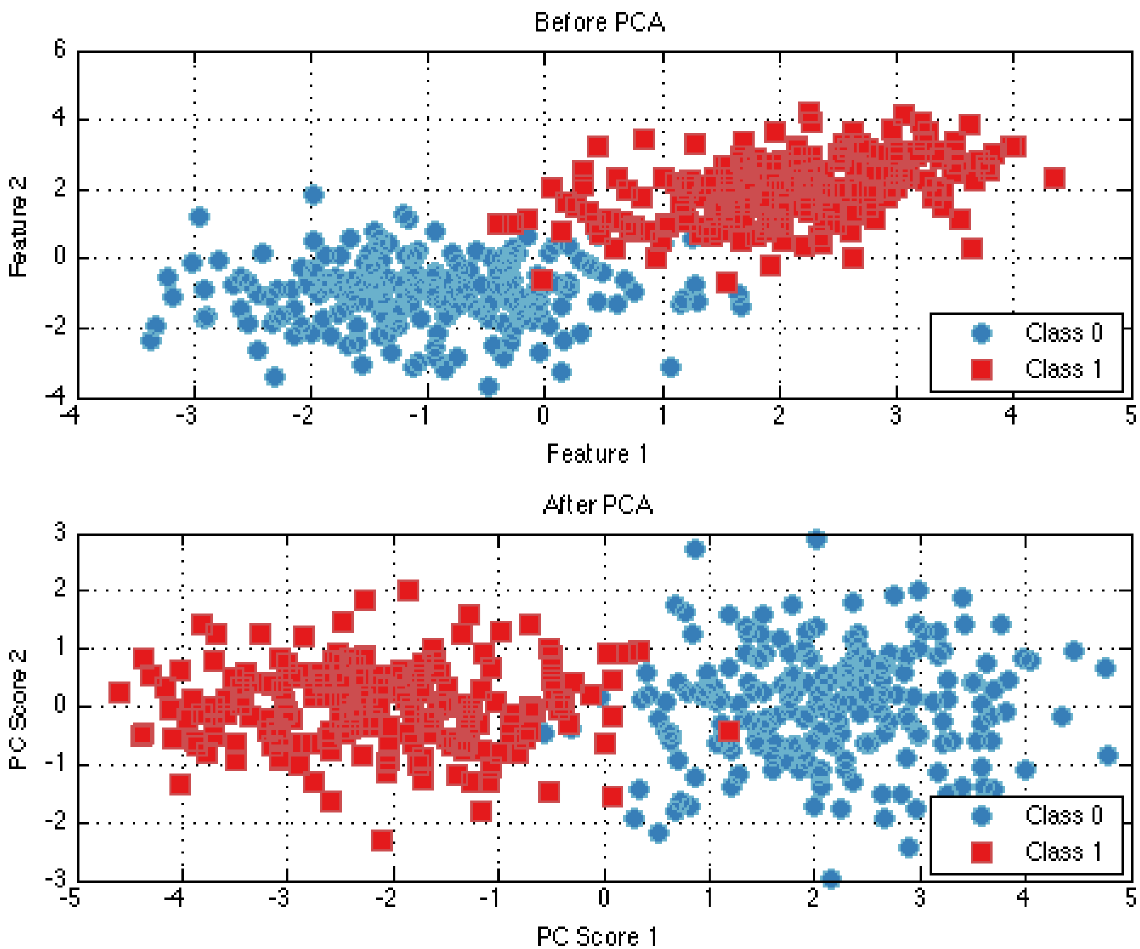
Overview

- ✓We learned to solve a system of linear equations
- ✓Solved Least Square problem (using normal equations)
- ✓**Some review on elementary linear algebra**
- ✓**Change of basis**
  - ✓Is it possible to define a new coordinate system?

What are the axis like?

Why we need to change the basis?

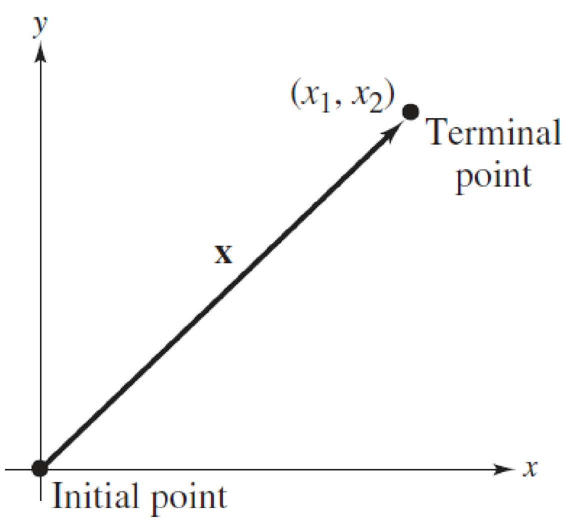


Vector spaces

Chapter 4 of book "elementary linear algebra"

Vectors in plane

- A vector is represented by a directed line segment
  - **initial point** at the origin
  - **terminal point** at (*x*1, *x*2),



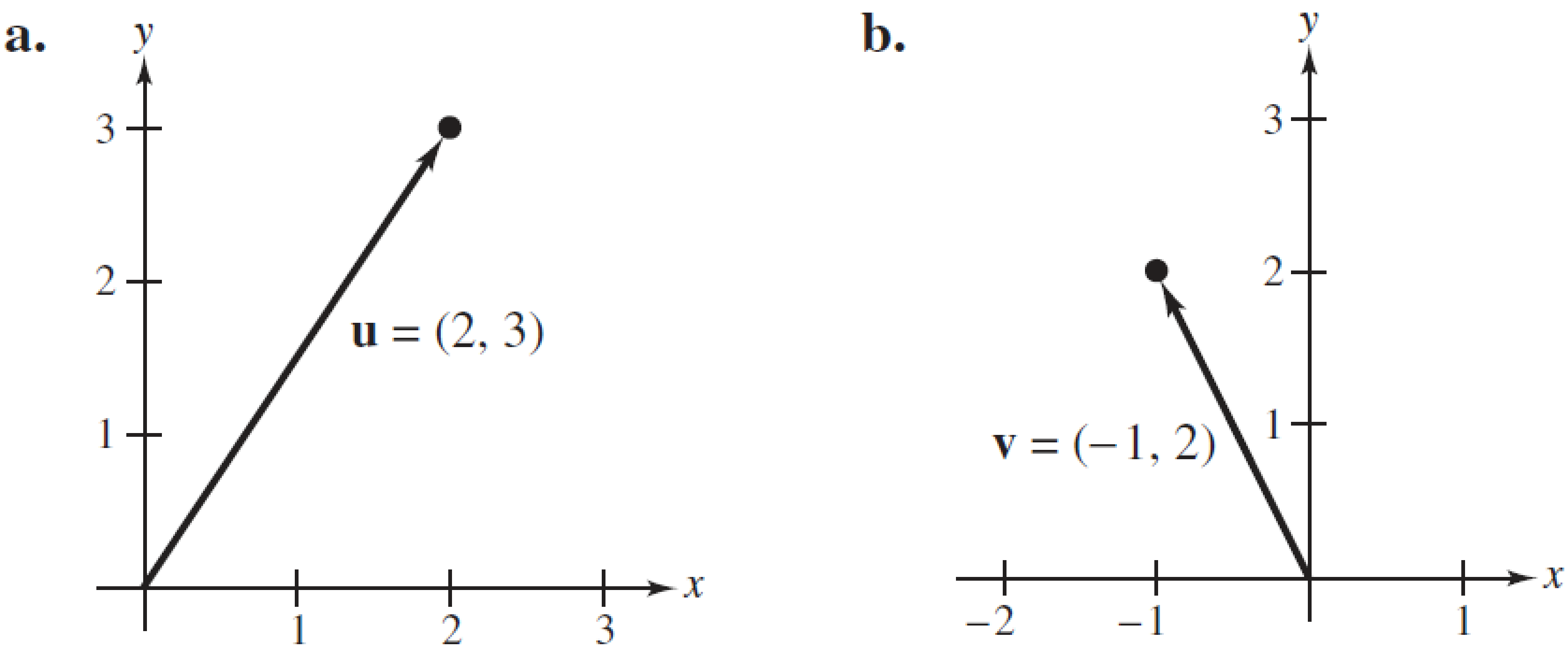
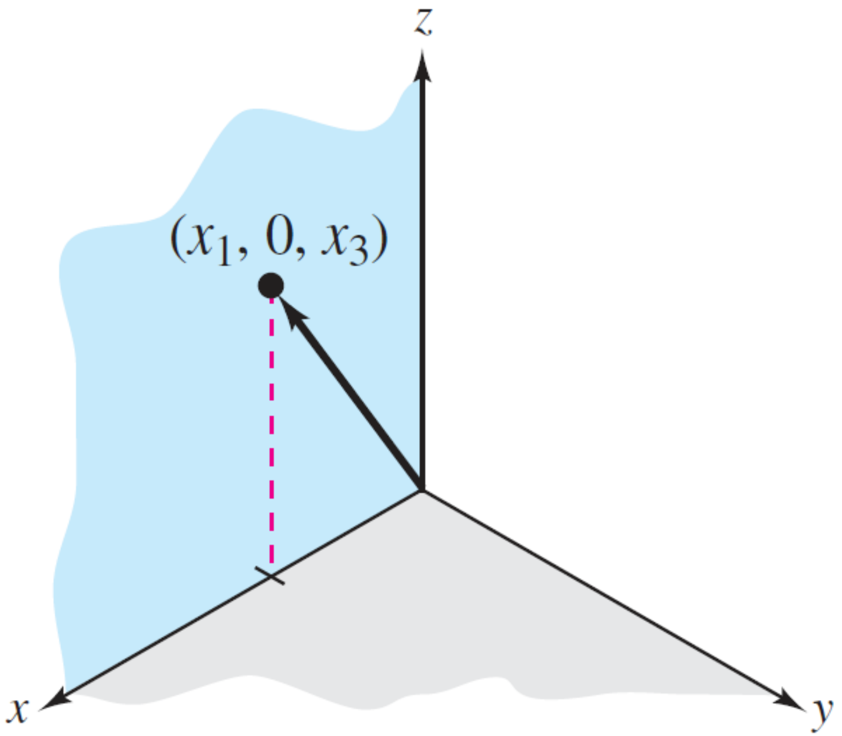
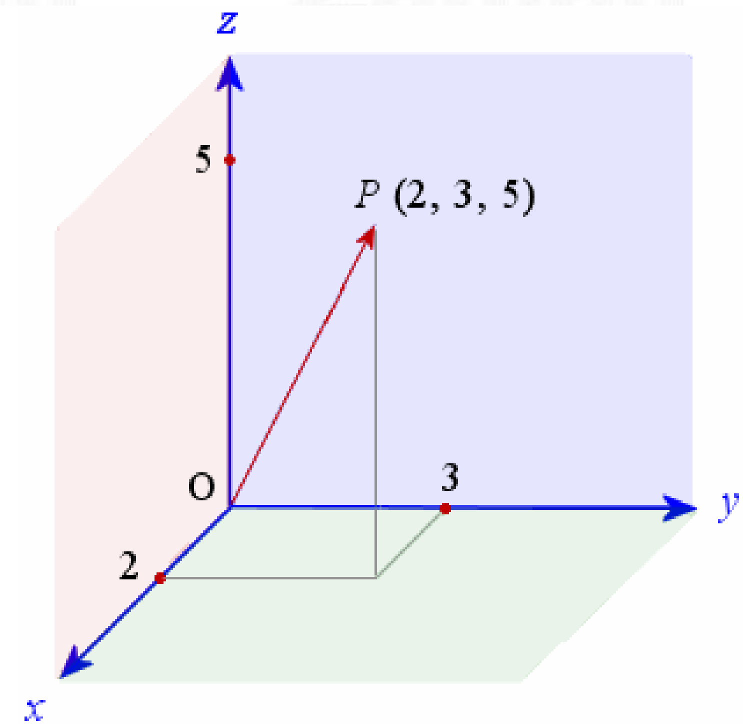


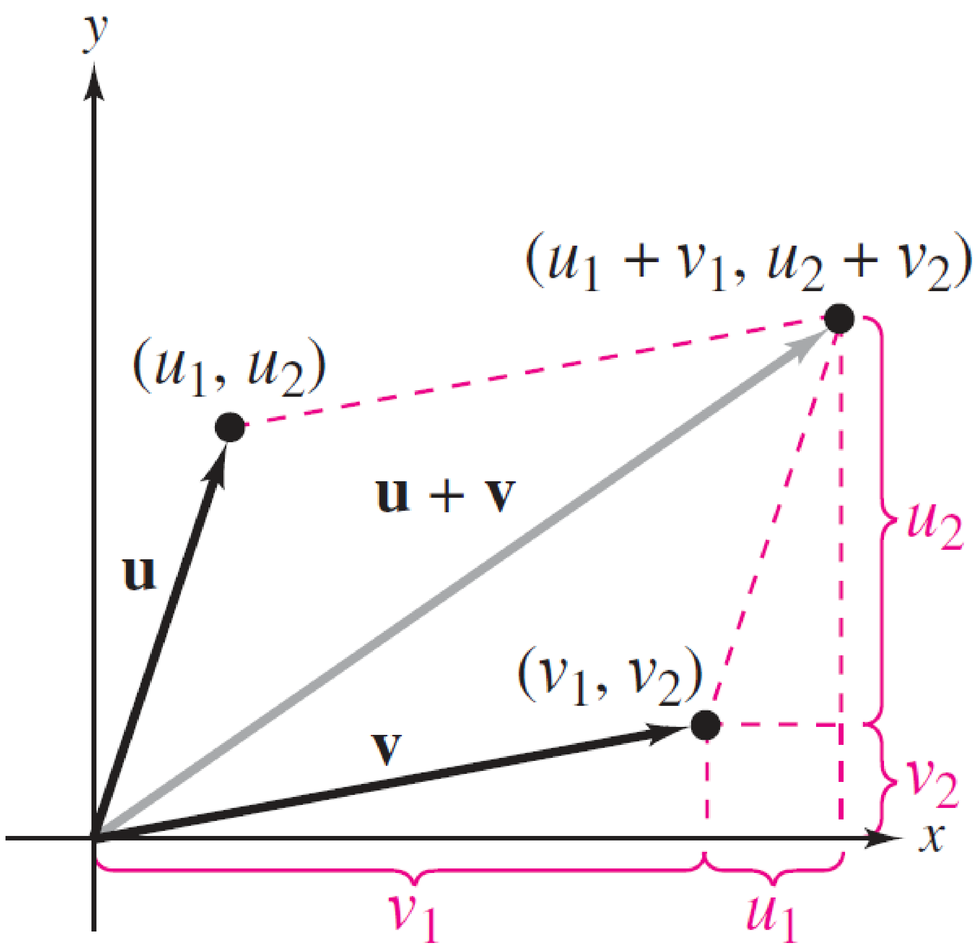
Figure 4.1

Examples of vectors in $\mathbb{R}^3$



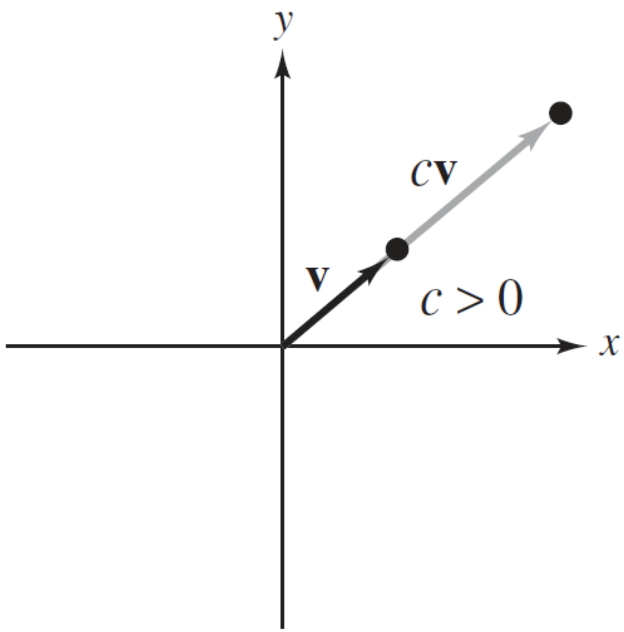


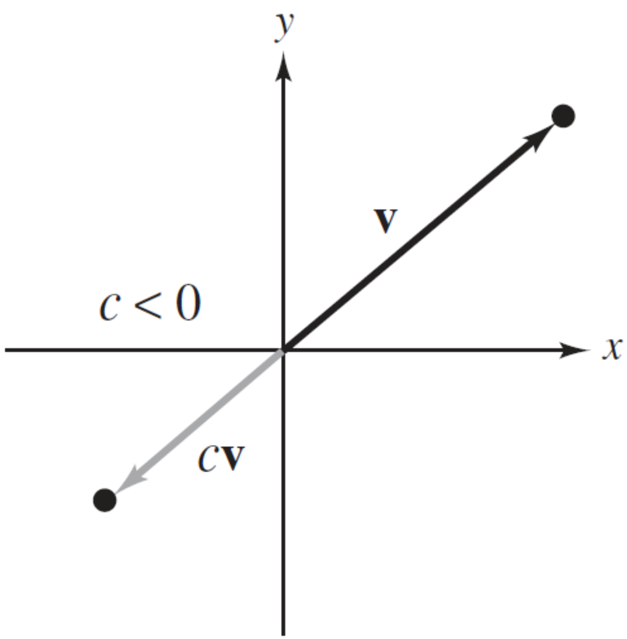
Adding two Vectors



Vector Addition

$$c\mathbf{v} = c(v_1, v_2) = (cv_1, cv_2)$$

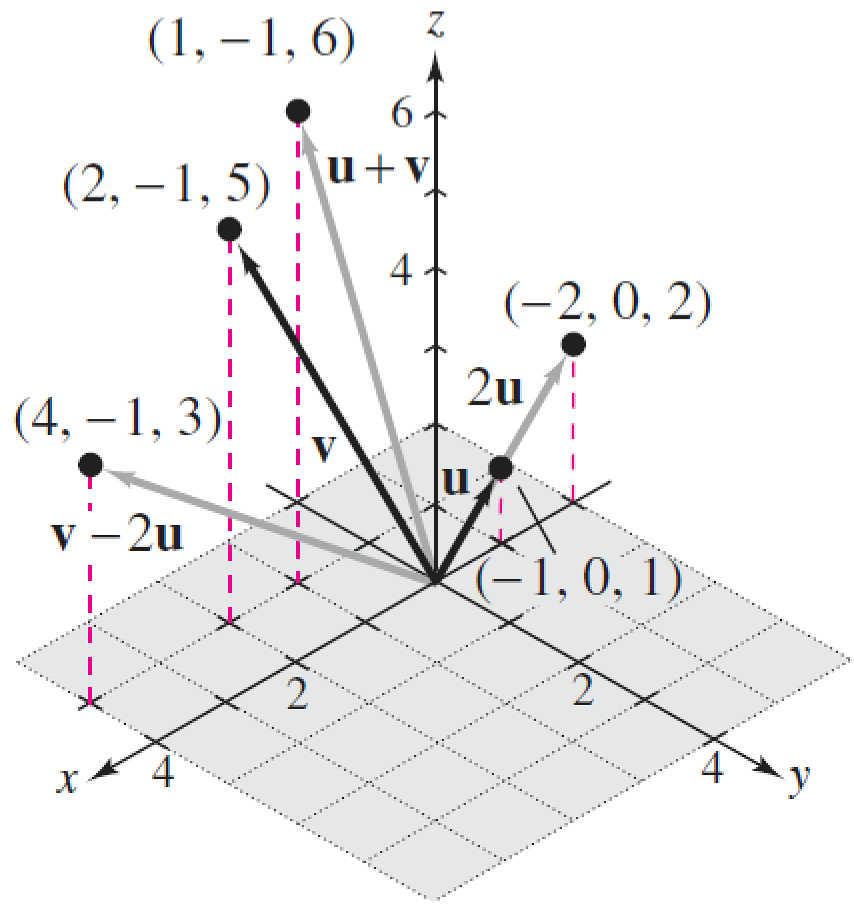




Properties of vector operations in $\mathbb{R}^2$

1. $\mathbf{u} + \mathbf{v}$ is a vector in the plane.
2. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
3. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$
4. $\mathbf{u} + \mathbf{0} = \mathbf{u}$
5. $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$
6. $c\mathbf{u}$ is a vector in the plane.
7. $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$
8. $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$
9. $c(d\mathbf{u}) = (cd)\mathbf{u}$
10. $1(\mathbf{u}) = \mathbf{u}$

Vectors in $\mathbb{R}^n$



R^1 = 1-space = set of all real numbers

R^2 = 2-space = set of all ordered pairs of real numbers

R^3 = 3-space = set of all ordered triples of real numbers

$\vdots$

R^n = n-space = set of all ordered n-tuples of real numbers

Properties of Vector Addition and Scalar Multiplication in $\mathbb{R}^n$

Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ be vectors in R^n, and let c and d be scalars.

1. $\mathbf{u} + \mathbf{v}$ is a vector in R^n.	Closure under addition
2. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$	Commutative property of addition
3. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$	Associative property of addition
4. $\mathbf{u} + \mathbf{0} = \mathbf{u}$	Additive identity property
5. $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$	Additive inverse property
6. $c\mathbf{u}$ is a vector in R^n.	Closure under scalar multiplication
7. $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$	Distributive property
8. $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$	Distributive property
9. $c(d\mathbf{u}) = (cd)\mathbf{u}$	Associative property of multiplication
10. $1(\mathbf{u}) = \mathbf{u}$	Multiplicative identity property

Linear Combination of Vectors

- a_1 = (−1, −2, −2), a_2 = (0, 1, 4), a_3 = (−1, 1, 2)
- Linear combination of a_1, a_2 and a_3 ($x_1, x_2, x_3 \in \mathbb{R}$)

$$z = x_1 a_1 + x_2 a_2 + x_3 a_3$$

- For $x_1 = 1, x_2 = 2, x_3 = -1$

$$z = a_1 + 2a_2 - a_3$$

$$z = (0, -1, 4)$$

Definition of a vector space

- We reviewed Properties of Vector Addition and Scalar Multiplication in $\mathbb{R}^{\boldsymbol{n}}$
- *Any* set that satisfies these properties (or **axioms**) is called a **vector space,** and the objects in the set are **vectors.**

Definition of a Vector Space

Let V be a set on which two operations (**vector addition** and **scalar multiplication**) are defined. If the listed axioms are satisfied for every $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ in V and every scalar (real number) c and d, then V is a **vector space.**

Addition:

1. $\mathbf{u} + \mathbf{v}$ is in V.	Closure under addition
2. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$	Commutative property
3. $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$	Associative property
4. V has a **zero vector 0** such that for every $\mathbf{u}$ in V, $\mathbf{u} + \mathbf{0} = \mathbf{u}$.	Additive identity
5. For every $\mathbf{u}$ in V, there is a vector in V denoted by $-\mathbf{u}$ such that $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$.	Additive inverse

Scalar Multiplication:

6. $c\mathbf{u}$ is in V.	Closure under scalar multiplication
7. $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$	Distributive property
8. $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$	Distributive property
9. $c(d\mathbf{u}) = (cd)\mathbf{u}$	Associative property
10. $1(\mathbf{u}) = \mathbf{u}$	Scalar identity

A vector space consists of four entities: a set of vectors, a set of scalars, and two operations.

Examples

- The set of all ordered pairs of real numbers $\mathbb{R}^2$ with the standard operations is a vector space.
  - **v** = (*v*1, *v*2)
- The set of all ordered pairs of real numbers $\mathbb{R}^n$ with the standard operations is a vector space.
  - **v** = (*v*1, *v*2, *v*3, . . . , *vn*)
- The set of all 2 × 3 matrices with the operations of matrix addition and scalar multiplication

Summary of Important Vector Spaces

R = set of all real numbers

R^2 = set of all ordered pairs

R^3 = set of all ordered triples

R^n = set of all n-tuples

$C(-\infty, \infty)$ = set of all continuous functions defined on the real number line

$C[a, b]$ = set of all continuous functions defined on a closed interval $[a, b]$, where $a \neq b$

P = set of all polynomials

P_n = set of all polynomials of degree $\leq n$ (together with the zero polynomial)

$M_{m,n}$ = set of all $m \times n$ matrices

$M_{n,n}$ = set of all $n \times n$ square matrices

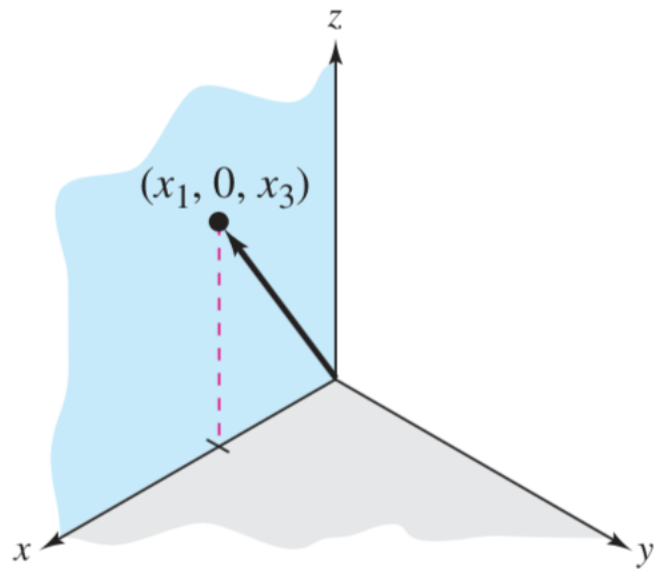
Why it is useful to define vector spaces?

Examples of what is not a vector space

- The set of integers
  - ½ * 1 = ½ (not an integer)
- The set of second-degree polynomials
  - $X^2 + (1+x-x^2) = 1+x$ (first degree)

Definition of a Subspace of a Vector Space

A nonempty subset W of a vector space V is a **subspace** of V when W is a vector space under the operations of addition and scalar multiplication defined in V.



The set $W = \{(x_1, 0, x_3): x_1 \text{ and } x_3 \text{ are real numbers}\}$ is a subspace of R^3

- Determine whether each subset is a subspace of R^2.
- **The set of points on the line $x + 2y = 0$**

$\mathbf{v}1 = (-2t1, t1)$ and $\mathbf{v}2 = (-2t2, t2)$

$\mathbf{v}1 + \mathbf{v}2 = (-2t1, t1) + (-2t2, t2) = (-2(t1 + t2), t1 + t2) = (-2t3, t3)$

- **The set of points on the line $x + 2y = 1$**

This subset of R^2 is ***not*** a subspace of R^2 because every subspace must contain the **zero vector** (0, 0), which is not on the line $x + 2y = 1$.

Definition of a Linear Combination of Vectors

A vector $\mathbf{v}$ in a vector space V is a **linear combination** of the vectors $\mathbf{u}_1$, $\mathbf{u}_2, \ldots, \mathbf{u}_k$ in V when $\mathbf{v}$ can be written in the form

$$\mathbf{v} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_k\mathbf{u}_k$$

where $c_1, c_2, \ldots, c_k$ are scalars.

Spanning sets

Definition of a Spanning Set of a Vector Space

Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ be a subset of a vector space V. The set S is a **spanning set** of V when *every* vector in V can be written as a linear combination of vectors in S. In such cases it is said that S **spans** V.

Spanning sets

Definition of a Spanning Set of a Vector Space

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Example:

$S = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$ spans R^3 because any vector $\mathbf{u} = (u_1, u_2, u_3)$ in R^3 can be written as

$$\mathbf{u} = u_1(1, 0, 0) + u_2(0, 1, 0) + u_3(0, 0, 1) = (u_1, u_2, u_3).$$

Example

$S = \{(1, 2, 3), (0, 1, 2), (-1, 0, 1)\}$

- *does* *not* span R^3 because $\mathbf{w} = (1, -2, 2)$ is in R^3 and cannot be expressed as a linear combination of the vectors in S.

Example

- Show that the set $S = \{(1, 2, 3), (0, 1, 2), (-2, 0, 1)\}$ spans R^3.

Example

- Show that the set $S = \{(1, 2, 3), (0, 1, 2), (-2, 0, 1)\}$ spans R^3.
- Find scalars c_1, c_2, and c_3 such that

$$
\begin{aligned}
(u_1, u_2, u_3) &= c_1(1, 2, 3) + c_2(0, 1, 2) + c_3(-2, 0, 1) \\
&= (c_1 - 2c_3,\ 2c_1 + c_2,\ 3c_1 + 2c_2 + c_3)
\end{aligned}
$$

Example

- Show that the set $S = \{(1, 2, 3), (0, 1, 2), (-2, 0, 1)\}$ spans R^3.
- Find scalars c_1, c_2, and c_3 such that

$$\begin{aligned}(u_1, u_2, u_3) &= c_1(1, 2, 3) + c_2(0, 1, 2) + c_3(-2, 0, 1) \\ &= (c_1 - 2c_3, 2c_1 + c_2, 3c_1 + 2c_2 + c_3)\end{aligned}$$

$$\begin{aligned} c_1 \phantom{{}+ 2c_2} - 2c_3 &= u_1 \\ 2c_1 + c_2 \phantom{{}- 2c_3} &= u_2 \\ 3c_1 + 2c_2 + c_3 &= u_3 \end{aligned}$$

Definition of the Span of a Set

If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is a set of vectors in a vector space V, then the **span of S** is the set of all linear combinations of the vectors in S,

$$\text{span}(S) = \{c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k : c_1, c_2, \ldots, c_k \text{ are real numbers}\}.$$

The span of S is denoted by

$$\text{span}(S) \quad \text{or} \quad \text{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}.$$

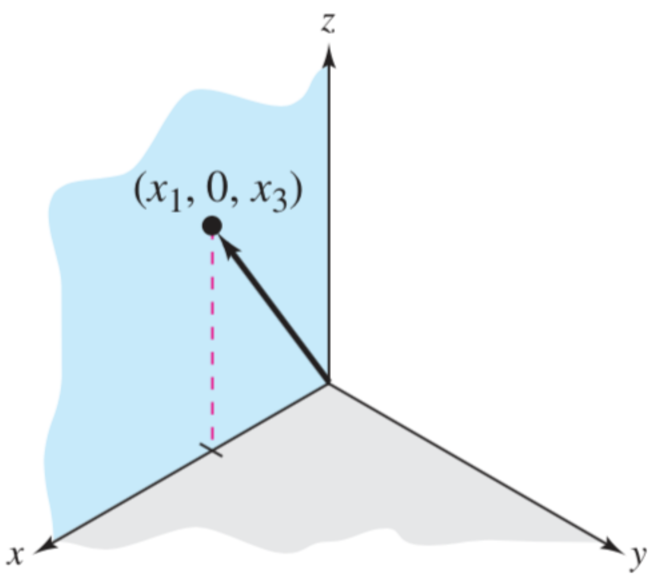
When $\text{span}(S) = V$, it is said that V is **spanned** by $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$, or that S **spans** V.

THEOREM 4.7 Span(S) Is a Subspace of V

If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is a set of vectors in a vector space V, then span(S) is a subspace of V. Moreover, span(S) is the smallest subspace of V that contains S, in the sense that every other subspace of V that contains S must contain span(S).

Example

S = {(1, 0,0) , (0,0,1)}   (x,y,z)



Definition of Linear Dependence and Linear Independence

A set of vectors $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ in a vector space V is **linearly independent** when the vector equation

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k = \mathbf{0}$$

has only the trivial solution

$$\boxed{c_1 = 0, c_2 = 0, \ldots, c_k = 0.}$$

If there are also nontrivial solutions, then S is **linearly dependent.**

THEOREM 4.8 A Property of Linearly Dependent Sets

A set $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$, $k \geq 2$, is linearly dependent if and only if at least one of the vectors $\mathbf{v}_i$ can be written as a linear combination of the other vectors in S.

Testing for Linear Independence

$$S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} = \{(1, 2, 3), (0, 1, 2), (-2, 0, 1)\}$$

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{0}$$

$$c_1(1, 2, 3) + c_2(0, 1, 2) + c_3(-2, 0, 1) = (0, 0, 0)$$

$$(c_1 - 2c_3, 2c_1 + c_2, 3c_1 + 2c_2 + c_3) = (0, 0, 0)$$

$$S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} = \{(1, 2, 3), (0, 1, 2), (-2, 0, 1)\}$$

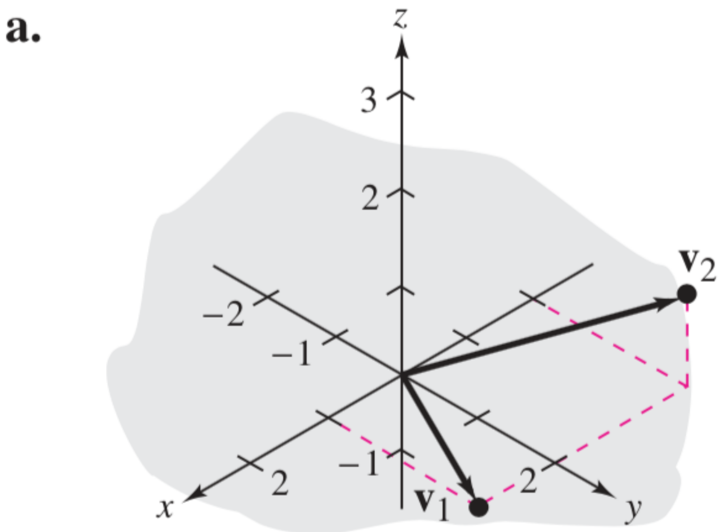
$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{0}$$

$$(c_1 - 2c_3, 2c_1 + c_2, 3c_1 + 2c_2 + c_3) = (0, 0, 0)$$

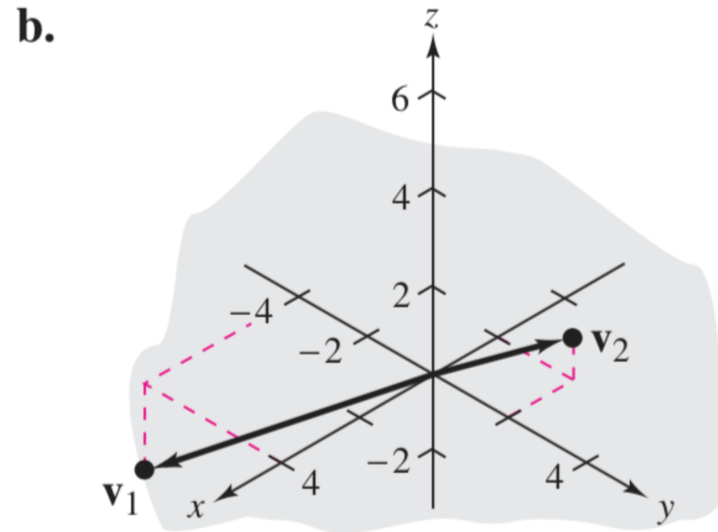
$$\begin{aligned} c_1 \phantom{{}+2c_2} &- 2c_3 = 0 \\ 2c_1 + c_2 \phantom{{}+c_3} &= 0 \\ 3c_1 + 2c_2 + c_3 &= 0 \end{aligned}$$

$$\begin{bmatrix} 1 & 0 & -2 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 2 & 1 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$c_1 = c_2 = c_3 = 0$. So, S is linearly independent



$S = \{(1, 2, 0), (-2, 2, 1)\}$
The set S is linearly independent.



$S = \{(4, -4, -2), (-2, 2, 1)\}$
The set S is linearly dependent.

Two vectors **u** and **v** in a vector space *V* are linearly dependent if and only if one is a scalar multiple of the other.

Basis for a vector space

Definition of Basis

A set of vectors $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ in a vector space V is a **basis** for V when the conditions below are true.

1. S spans V. **2.** S is linearly independent.

Example

$S = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$ is a basis for R^3

S is the **standard basis** for R^3. This can be generalized to *n*-space

Example: A Nonstandard Basis for R^2

- $S = \{(1, 1), (1, -1)\}$

- **Homework:** prove it!

THEOREM 4.11 Number of Vectors in a Basis

If a vector space V has one basis with n vectors, then every basis for V has n vectors.

THEOREM 4.10 Bases and Linear Dependence

If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a basis for a vector space V, then every set containing more than n vectors in V is linearly dependent.

Only one of the conditions will suffice!

THEOREM 4.12 Basis Tests in an *n*-Dimensional Space

Let V be a vector space of dimension n.

1. If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a linearly independent set of vectors in V, then S is a basis for V.
2. If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ spans V, then S is a basis for V.

THEOREM 4.9 Uniqueness of Basis Representation

If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a basis for a vector space V, then every vector in V can be written in one and only one way as a linear combination of vectors in S.

Proof: Assume that a vector u has two representations:

$$\begin{cases} \mathbf{u} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n \\ \mathbf{u} = b_1\mathbf{v}_1 + b_2\mathbf{v}_2 + \cdots + b_n\mathbf{v}_n \end{cases}$$

$$\Rightarrow \mathbf{u} - \mathbf{u} = (c_1 - b_1)\mathbf{v}_1 + (c_2 - b_2)\mathbf{v}_2 + \cdots + (c_n - b_n)\mathbf{v}_n = \mathbf{0}$$

$$\Rightarrow c_1 - b_1 = 0, \quad c_2 - b_2 = 0, \quad \ldots, \quad c_n - b_n = 0$$

which means that $c_i = b_i$ for all $i = 1, 2, \ldots, n$, and **u** has only one representation for the basis S.

THEOREM 4.9 Uniqueness of Basis Representation

If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a basis for a vector space V, then every vector in V can be written in one and only one way as a linear combination of vectors in S.

n: dimension of v

Coordinates and Change of Basis

- In Theorem 4.9, you saw that if *B* is a basis for a vector space *V*, then every vector **x** in *V* can be expressed in one and only one way as a linear combination of vectors in *B*.
- The coefficients in the linear combination are the **coordinates of x relative to *B*.**

$$\boxed{S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} = \{(1, 2, 3), (0, 1, 2), (-2, 0, 1)\}}$$

How can we find the coordinate of point P with regard to this new basis?

Coordinate Representation Relative to a Basis

Let $B = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be an ordered basis for a vector space V and let $\mathbf{x}$ be a vector in V such that

$$\mathbf{x} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n.$$

The scalars $c_1, c_2, \ldots, c_n$ are the **coordinates of x relative to the basis *B*.** The **coordinate matrix** (or **coordinate vector**) **of x relative to *B*** is the column matrix in R^n whose components are the coordinates of $\mathbf{x}$.

$$[\mathbf{x}]_B = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

- For the vector **x** = $(x_1, x_2, \ldots, x_n)$, the x_i's are the coordinates of **x** *relative to the standard basis S for* R^n.

$$[\mathbf{x}]_S = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

Find the coordinate matrix of $x = (-2, 1, 3)$ in R^3 relative to the standard basis
$S = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$.

Find the coordinate matrix of x = (−2, 1, 3) in R^3 relative to the standard basis
$S = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$.

$$[\mathbf{x}]_S = \begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix}.$$

Finding a Coordinate Matrix Relative to a Standard Basis

The coordinate matrix of **x** in R^2 relative to the (nonstandard) ordered basis B = {**v**1, **v**2} = {(1, 0), (1, 2)} is

$$[\mathbf{x}]_B = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

Find the coordinate matrix of **x** relative to the standard basis B' = {**u** , **u** } = {(1, 0), (0, 1)}.

The coordinate matrix of **x** in R^2 relative to the (nonstandard) ordered basis B = {**v**1, **v**2} = {(1, 0), (1, 2)} is

$$[\mathbf{x}]_B = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

Find the coordinate matrix of **x** relative to the standard basis B' = {**u** , **u** } = {(1, 0), (0, 1)}.

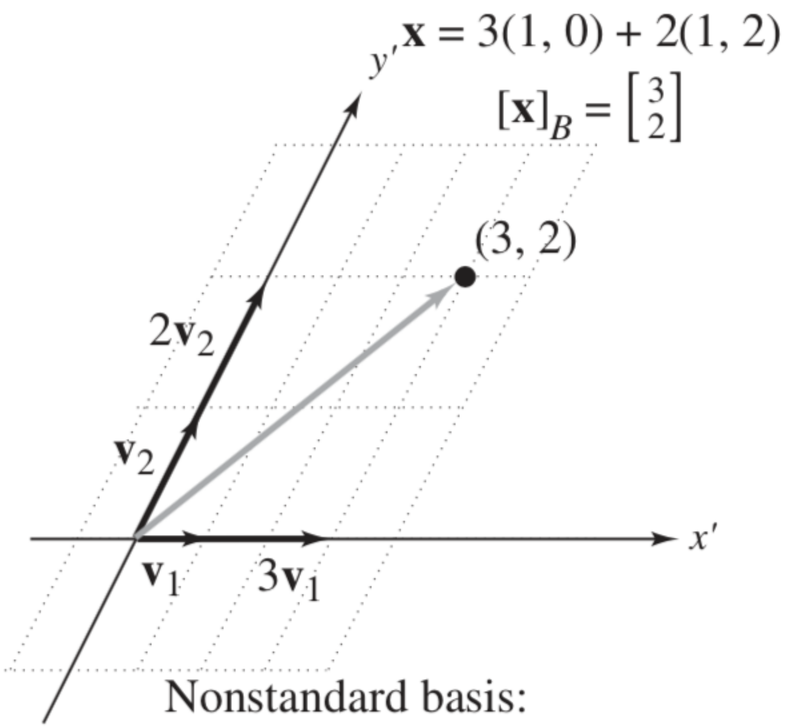
solution

The coordinate matrix of $\mathbf{x}$ relative to B is $[\mathbf{x}]_B = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$, so

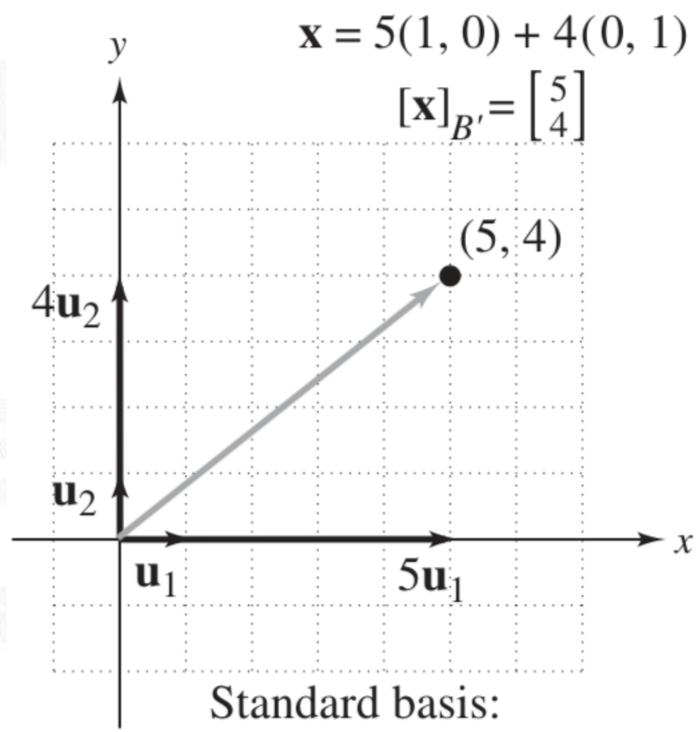
$$\mathbf{x} = 3\mathbf{v}_1 + 2\mathbf{v}_2 = 3(1, 0) + 2(1, 2) = (5, 4) = 5(1, 0) + 4(0, 1).$$

It follows that the coordinate matrix of $\mathbf{x}$ relative to B' is

$$[\mathbf{x}]_{B'} = \begin{bmatrix} 5 \\ 4 \end{bmatrix}.$$



Nonstandard basis:
$B = \{(1, 0), (1, 2)\}$



Standard basis:
$B' = \{(1, 0), (0, 1)\}$

Finding a Coordinate Matrix Relative to a Nonstandard Basis

- Find the coordinate matrix of x = (1, 2, −1) in R^3 relative to the (nonstandard) basis

$$B' = \{u1, u2, u3\} = \{(1, 0, 1), (0, -1, 2), (2, 3, -5)\}$$

$$x = c_1 u_1 + c_2 u_2 + c_3 u_3$$

$$(1, 2, -1) = c_1(1, 0, 1) + c_2(0, -1, 2) + c_3(2, 3, -5)$$

$$\mathbf{x} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + c_3\mathbf{u}_3$$
$$(1, 2, -1) = c_1(1, 0, 1) + c_2(0, -1, 2) + c_3(2, 3, -5)$$

$$\mathbf{x} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + c_3\mathbf{u}_3$$
$$(1, 2, -1) = c_1(1, 0, 1) + c_2(0, -1, 2) + c_3(2, 3, -5)$$

$$\begin{aligned} c_1 \quad\quad\; + 2c_3 &= \;\;1 \\ -c_2 + 3c_3 &= \;\;2 \\ c_1 + 2c_2 - 5c_3 &= -1 \end{aligned}$$

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & 3 \\ 1 & 2 & -5 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

$$\mathbf{x} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + c_3\mathbf{u}_3$$
$$(1, 2, -1) = c_1(1, 0, 1) + c_2(0, -1, 2) + c_3(2, 3, -5)$$

$$\begin{aligned} c_1 \qquad\qquad + 2c_3 &= 1 \\ -c_2 + 3c_3 &= 2 \\ c_1 + 2c_2 - 5c_3 &= -1 \end{aligned}$$

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & 3 \\ 1 & 2 & -5 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

The solution of this system is $c_1 = 5$, $c_2 = -8$, and $c_3 = -2$. So,

$$\mathbf{x} = 5(1, 0, 1) + (-8)(0, -1, 2) + (-2)(2, 3, -5)$$

$$[\mathbf{x}]_{B'} = \begin{bmatrix} 5 \\ -8 \\ -2 \end{bmatrix}$$

Homework

- Given a set of N points in R2 $B = \{\mathbf{v}1, \mathbf{v}2\} = \{(1, 0), (1, 2)\}$, transfer them to their coordinates with standard basis and visualize the two set
- Given the standard coordinates, how can you transfer them back to the space with basis $B = \{\mathbf{v}1, \mathbf{v}2\} = \{(1, 0), (1, 2)\}$
  - See Section 4.7 of the book for every arbitrary B and B'

CHANGE OF BASIS IN R^n

- Finding a Coordinate Matrix Relative to another Basis (standard or non-standard)
  - Given the coordinates of a vector relative to a basis B
  - Find the coordinates relative to another basis B'

- General formulation?

If B is the standard Basis

$$\underbrace{\begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & 3 \\ 1 & 2 & -5 \end{bmatrix}}_{P}\underbrace{\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}}_{[\mathbf{x}]_{B'}} = \underbrace{\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}}_{[\mathbf{x}]_{B}}$$

General formula

$$\underbrace{\begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & 3 \\ 1 & 2 & -5 \end{bmatrix}}_{P} \underbrace{\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}}_{[\mathbf{x}]_{B'}} = \underbrace{\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}}_{[\mathbf{x}]_{B}}$$

Transition matrix
from B′ to B

coordinate matrix
of **x** relative to *B'*

coordinate matrix
of **x** relative to *B*

Change of Basis from B to B’

$$P[\mathbf{x}]_{B'} = [\mathbf{x}]_B$$

Change of Basis from B to B’

$$P[\mathbf{x}]_{B'} = [\mathbf{x}]_B$$

$$\boxed{[\mathbf{x}]_{B'} = P^{-1}[\mathbf{x}]_B}$$

Example

$$\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \underbrace{\begin{bmatrix} -1 & 4 & 2 \\ 3 & -7 & -3 \\ 1 & -2 & -1 \end{bmatrix}}_{P^{-1}} \underbrace{\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}}_{[\mathbf{x}]_B} = \underbrace{\begin{bmatrix} 5 \\ -8 \\ -2 \end{bmatrix}}_{[\mathbf{x}]_{B'}}$$

Generalization

$$B = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\} \text{ and } B' = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$$

$$[\mathbf{x}]_B = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} \text{ and } [\mathbf{x}]_{B'} = \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_n \end{bmatrix}$$

$$[\mathbf{x}]_{B'} = P^{-1}[\mathbf{x}]_B.$$

Coordinate matrix of $\mathbf{x}$ relative to B'

Transition matrix from B to B'

Coordinate matrix of $\mathbf{x}$ relative to B

Generalization

$$B = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\} \text{ and } B' = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$$

$$[\mathbf{x}]_B = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} \text{ and } [\mathbf{x}]_{B'} = \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_n \end{bmatrix}$$

$$\underbrace{[\mathbf{x}]_{B'}}_{\text{Coordinate matrix of } \mathbf{x} \text{ relative to } B'} = \underbrace{P^{-1}}_{\text{Transition matrix from } B \text{ to } B'} \underbrace{[\mathbf{x}]_B}_{\text{Coordinate matrix of } \mathbf{x} \text{ relative to } B}.$$

THEOREM 4.20 The Inverse of a Transition Matrix

If P is the transition matrix from a basis B' to a basis B in R^n, then P is invertible and the transition matrix from B to B' is P^{-1}.

LEMMA

Let $B = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ and $B' = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$ be two bases for a vector space V. If

$$\begin{aligned}
\mathbf{v}_1 &= c_{11}\mathbf{u}_1 + c_{21}\mathbf{u}_2 + \cdots + c_{n1}\mathbf{u}_n \\
\mathbf{v}_2 &= c_{12}\mathbf{u}_1 + c_{22}\mathbf{u}_2 + \cdots + c_{n2}\mathbf{u}_n \\
&\vdots \\
\mathbf{v}_n &= c_{1n}\mathbf{u}_1 + c_{2n}\mathbf{u}_2 + \cdots + c_{nn}\mathbf{u}_n
\end{aligned}$$

then the transition matrix from B to B' is

$$Q = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & & \vdots \\ c_{n1} & c_{n2} & \cdots & c_{nn} \end{bmatrix}.$$

LEMMA

Let $B = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ and $B' = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$ be two bases for a vector space V. If

$$\begin{aligned}
\mathbf{v}_1 &= c_{11}\mathbf{u}_1 + c_{21}\mathbf{u}_2 + \cdots + c_{n1}\mathbf{u}_n \\
\mathbf{v}_2 &= c_{12}\mathbf{u}_1 + c_{22}\mathbf{u}_2 + \cdots + c_{n2}\mathbf{u}_n \\
&\vdots \\
\mathbf{v}_n &= c_{1n}\mathbf{u}_1 + c_{2n}\mathbf{u}_2 + \cdots + c_{nn}\mathbf{u}_n
\end{aligned}$$

then the transition matrix from B to B' is

$$Q = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & & \vdots \\ c_{n1} & c_{n2} & \cdots & c_{nn} \end{bmatrix}.$$

$$\boxed{Q[\mathbf{v}]_B = [\mathbf{v}]_{B'}}$$

Proof (read it at home)

Let $\mathbf{v} = d_1\mathbf{v}_1 + d_2\mathbf{v}_2 + \cdot \cdot \cdot + d_n\mathbf{v}_n$ be an arbitrary vector in V. The coordinate matrix of $\mathbf{v}$ with respect to the basis B is

$$[\mathbf{v}]_B = \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_n \end{bmatrix}.$$

Proof (cont.)

Let $\mathbf{v} = d_1\mathbf{v}_1 + d_2\mathbf{v}_2 + \cdots + d_n\mathbf{v}_n$ be an arbitrary vector in V. The coordinate matrix of $\mathbf{v}$ with respect to the basis B is

$$[\mathbf{v}]_B = \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_n \end{bmatrix}.$$

Then you have

$$\boxed{Q[\mathbf{v}]_B} = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & & \vdots \\ c_{n1} & c_{n2} & \cdots & c_{nn} \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_n \end{bmatrix} = \begin{bmatrix} c_{11}d_1 + c_{12}d_2 + \cdots + c_{1n}d_n \\ c_{21}d_1 + c_{22}d_2 + \cdots + c_{2n}d_n \\ \vdots \\ c_{n1}d_1 + c_{n2}d_2 + \cdots + c_{nn}d_n \end{bmatrix}.$$

Proof (cont.)

On the other hand,

$$
\begin{aligned}
\mathbf{v} &= d_1\mathbf{v}_1 + d_2\mathbf{v}_2 + \cdots + d_n\mathbf{v}_n \\
&= d_1(c_{11}\mathbf{u}_1 + c_{21}\mathbf{u}_2 + \cdots + c_{n1}\mathbf{u}_n) + d_2(c_{12}\mathbf{u}_1 + c_{22}\mathbf{u}_2 + \cdots + c_{n2}\mathbf{u}_n) + \cdots \\
&\qquad + d_n(c_{1n}\mathbf{u}_1 + c_{2n}\mathbf{u}_2 + \cdots + c_{nn}\mathbf{u}_n) \\
&= (d_1c_{11} + d_2c_{12} + \cdots + d_nc_{1n})\mathbf{u}_1 + (d_1c_{21} + d_2c_{22} + \cdots + d_nc_{2n})\mathbf{u}_2 + \cdots \\
&\qquad + (d_1c_{n1} + d_2c_{n2} + \cdots + d_nc_{nn})\mathbf{u}_n
\end{aligned}
$$

Proof (cont.)

On the other hand,

$$\begin{aligned}
\mathbf{v} &= d_1\mathbf{v}_1 + d_2\mathbf{v}_2 + \cdots + d_n\mathbf{v}_n \\
&= d_1(c_{11}\mathbf{u}_1 + c_{21}\mathbf{u}_2 + \cdots + c_{n1}\mathbf{u}_n) + d_2(c_{12}\mathbf{u}_1 + c_{22}\mathbf{u}_2 + \cdots + c_{n2}\mathbf{u}_n) + \cdots \\
&\qquad + d_n(c_{1n}\mathbf{u}_1 + c_{2n}\mathbf{u}_2 + \cdots + c_{nn}\mathbf{u}_n) \\
&= (d_1c_{11} + d_2c_{12} + \cdots + d_nc_{1n})\mathbf{u}_1 + (d_1c_{21} + d_2c_{22} + \cdots + d_nc_{2n})\mathbf{u}_2 + \cdots \\
&\qquad + (d_1c_{n1} + d_2c_{n2} + \cdots + d_nc_{nn})\mathbf{u}_n
\end{aligned}$$

which implies

$$[\mathbf{v}]_{B'} = \begin{bmatrix} c_{11}d_1 + c_{12}d_2 + \cdots + c_{1n}d_n \\ c_{21}d_1 + c_{22}d_2 + \cdots + c_{2n}d_n \\ \vdots \\ c_{n1}d_1 + c_{n2}d_2 + \cdots + c_{nn}d_n \end{bmatrix}.$$

Proof (cont.)

On the other hand,

$$\begin{aligned}\mathbf{v} &= d_1\mathbf{v}_1 + d_2\mathbf{v}_2 + \cdot\cdot\cdot + d_n\mathbf{v}_n \\ &= d_1(c_{11}\mathbf{u}_1 + c_{21}\mathbf{u}_2 + \cdot\cdot\cdot + c_{n1}\mathbf{u}_n) + d_2(c_{12}\mathbf{u}_1 + c_{22}\mathbf{u}_2 + \cdot\cdot\cdot + c_{n2}\mathbf{u}_n) + \cdot\cdot\cdot \\ &\qquad + d_n(c_{1n}\mathbf{u}_1 + c_{2n}\mathbf{u}_2 + \cdot\cdot\cdot + c_{nn}\mathbf{u}_n) \\ &= (d_1c_{11} + d_2c_{12} + \cdot\cdot\cdot + d_nc_{1n})\mathbf{u}_1 + (d_1c_{21} + d_2c_{22} + \cdot\cdot\cdot + d_nc_{2n})\mathbf{u}_2 + \cdot\cdot\cdot \\ &\qquad + (d_1c_{n1} + d_2c_{n2} + \cdot\cdot\cdot + d_nc_{nn})\mathbf{u}_n\end{aligned}$$

which implies

$$[\mathbf{v}]_{B'} = \begin{bmatrix} c_{11}d_1 + c_{12}d_2 + \cdot\cdot\cdot + c_{1n}d_n \\ c_{21}d_1 + c_{22}d_2 + \cdot\cdot\cdot + c_{2n}d_n \\ \vdots \qquad \vdots \qquad \vdots \\ c_{n1}d_1 + c_{n2}d_2 + \cdot\cdot\cdot + c_{nn}d_n \end{bmatrix}.$$

So, $Q[\mathbf{v}]_B = [\mathbf{v}]_{B'}$ and you can conclude that Q is the transition matrix from B to B'.

THEOREM 4.21 Transition Matrix from B to B'

Let

$$B = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\} \quad \text{and} \quad B' = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$$

be two bases for R^n. Then the transition matrix P^{-1} from B to B' can be found by using Gauss-Jordan elimination on the $n \times 2n$ matrix $[B' \quad B]$, as shown below.

$$[B' \quad B] \longrightarrow [I_n \quad P^{-1}]$$

Example of finding P (B → B')

$$B = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\} \quad \text{and} \quad B' = \{(1, 0, 1), (0, -1, 2), (2, 3, -5)\}$$

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ and } B' = \begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & 3 \\ 1 & 2 & -5 \end{bmatrix}$$

Example of finding P (B → B'): Solution

- Rewrite $[B' \quad B]$ as $[I_3 \quad P^{-1}]$ using Gauss-Jordan Elimination

$$\begin{bmatrix} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & -1 & 3 & 0 & 1 & 0 \\ 1 & 2 & -5 & 0 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & -1 & 4 & 2 \\ 0 & 1 & 0 & 3 & -7 & -3 \\ 0 & 0 & 1 & 1 & -2 & -1 \end{bmatrix}$$

Example of finding P (B → B'): Solution

- Rewrite $[B' \;\; B]$ as $[I_3 \;\; P^{-1}]$ using Gauss-Jordan Elimination

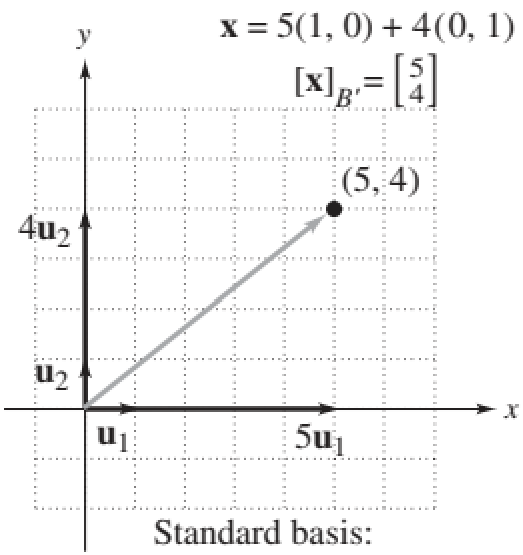
$$\begin{bmatrix} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & -1 & 3 & 0 & 1 & 0 \\ 1 & 2 & -5 & 0 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & -1 & 4 & 2 \\ 0 & 1 & 0 & 3 & -7 & -3 \\ 0 & 0 & 1 & 1 & -2 & -1 \end{bmatrix}$$

From this, you can conclude that the transition matrix from B to B' is

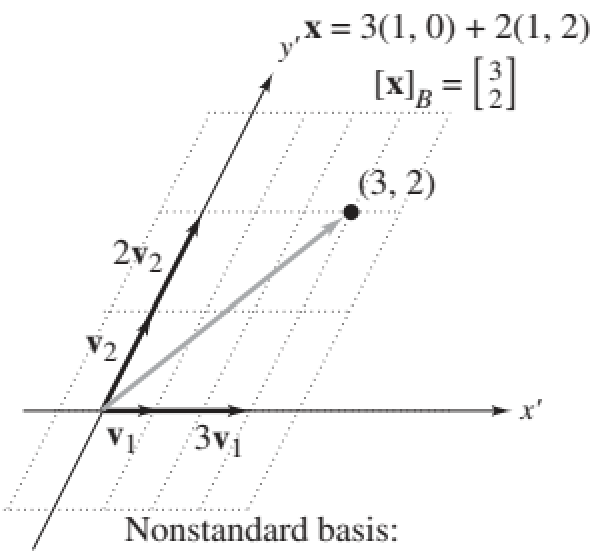
$$P^{-1} = \begin{bmatrix} -1 & 4 & 2 \\ 3 & -7 & -3 \\ 1 & -2 & -1 \end{bmatrix}.$$

Summary

- Vector space
- Basis for a vector space
- Coordinates and Change of basis
  - **Not necessarily orthogonal**

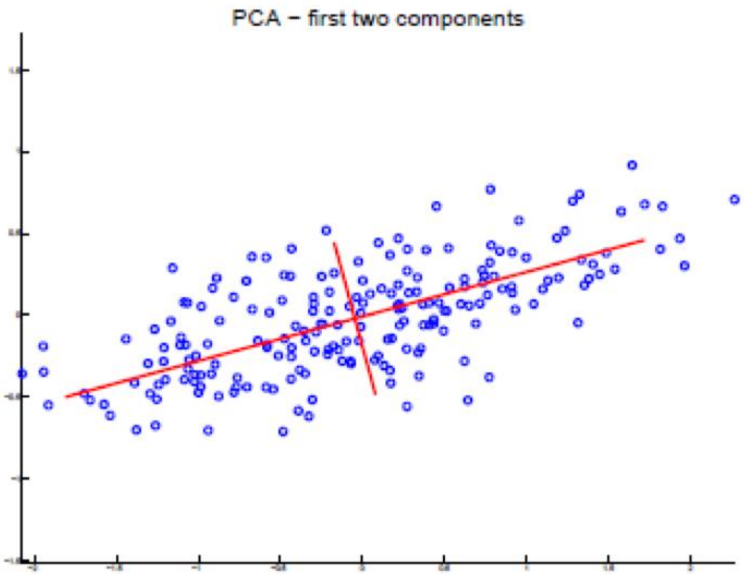


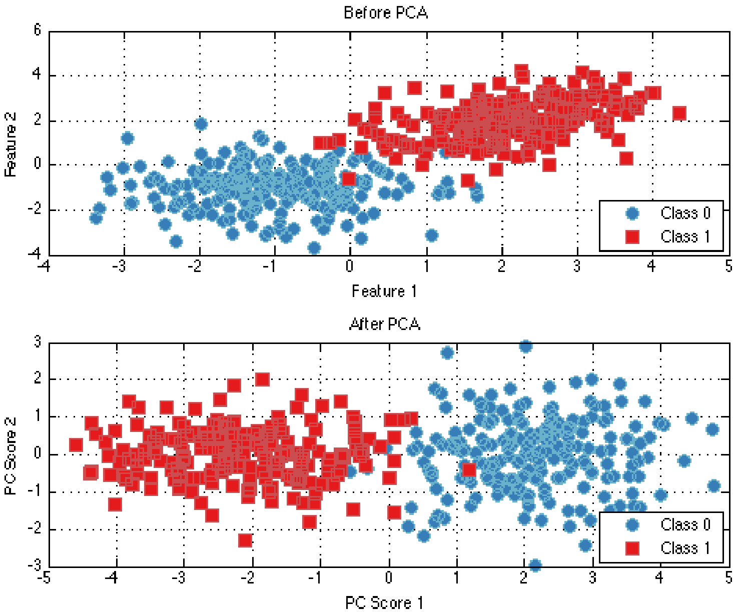
Standard basis:
$B' = \{(1, 0), (0, 1)\}$



Nonstandard basis:
$B = \{(1, 0), (1, 2)\}$

Where can we use this?





Any Question?