Computational Data Mining

Part 11: Eigenvalues and Eigenvectors

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Outline

- Eigenvalue and eigenvectors
- Diagonalization

Eigenvalue problem

the eigenvalue problem is one of the most important problems in linear algebra. Its central question is:

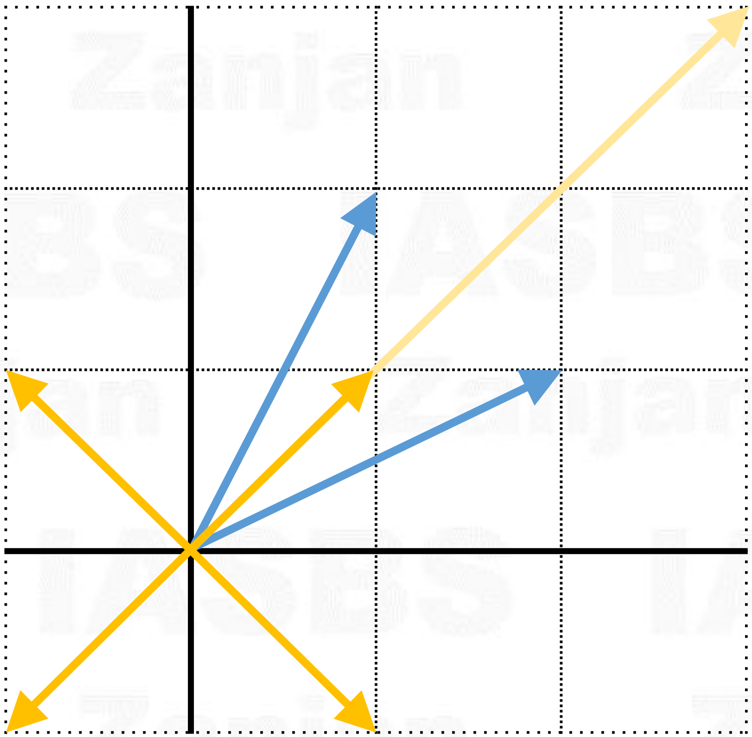
"when **A** is an n×n matrix, do nonzero vectors **x** in $\mathbf{R}^n$ exist such that **Ax** is a scalar multiple of **x**?"

$$\mathbf{Ax} = \lambda \mathbf{x}$$

Definitions

Let **A** be an n×n matrix. The scalar **λ** is an eigenvalue of **A** when there is a nonzero vector x such that $\mathbf{Ax} = \lambda\mathbf{x}$. The vector **x** is an eigenvector of **A** corresponding to λ.

Eigenvalue problem



$$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \lambda \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \; or \; \begin{bmatrix} 2 \\ 2 \end{bmatrix} \qquad \lambda = 3$$

$$\mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \; or \; \begin{bmatrix} -2 \\ 2 \end{bmatrix} \qquad \lambda = 1$$

$$\mathbf{A}(c\mathbf{x}) = c(\mathbf{A}\mathbf{x}) = c(\lambda\mathbf{x}) = \lambda(c\mathbf{x})$$

$$\mathbf{A}(\mathbf{x}_1 + \mathbf{x}_2) = \lambda(\mathbf{x}_1 + \mathbf{x}_2)$$

If **A** is an n×n matrix with an eigenvalue λ, then the set of all eigenvectors of λ, together with the zero vector is a subspace of R^n. This subspace is the **eigenspace** of **λ**.

Eigenvalue problem

Finding eigenvalue and eigenvectors

To find the eigenvalues and eigenvectors of an n×n matrix A, let I be the n×n identity matrix.

$$\mathbf{Ax} = \lambda\mathbf{x} \iff \mathbf{0} = \lambda\mathbf{x} - \mathbf{Ax}$$

$$\iff \mathbf{0} = \lambda\mathbf{I}_n\mathbf{x} - \mathbf{Ax}$$

$$\iff \mathbf{0} = \mathbf{x}(\lambda\mathbf{I}_n - \mathbf{A})$$

$$det(\lambda\mathbf{I} - \mathbf{A}) = \mathbf{0}$$

The equation $det(\lambda\mathbf{I} - \mathbf{A})=0$ is the **characteristic equation** of **A**. and when expanded in **polynomial** form is the characteristic polynomial

$$|\lambda I - A| = \lambda^n + c_{n-1}\lambda^{n-1} + \cdots + c_2\lambda^2 + c_1\lambda + c_0$$

Eigenvalue problem

Finding eigenvalue and eigenvectors

We will find the eigenvalues and corresponding eigenvectors of following matrix:

$$\mathbf{A} = \begin{bmatrix} 2 & -12 \\ 1 & -5 \end{bmatrix}$$

The characteristic polynomial of A is

$$|\lambda \mathbf{I} - \mathbf{A}| = \begin{vmatrix} \lambda - 2 & 12 \\ -1 & \lambda + 5 \end{vmatrix} = \lambda^2 + 3\lambda - 10 + 12$$

$$= (\lambda + 1)(\lambda + 2)$$

$$\lambda_1 = -1$$

$$\lambda_2 = -2$$

To find the corresponding eigenvectors, we solve the homogeneous linear system represented by $(\lambda\mathbf{I} - \mathbf{A})\mathbf{x} = 0$

$$-1\mathbf{I} - \mathbf{A} = \begin{bmatrix} -1 - 2 & 12 \\ -1 & -1 + 5 \end{bmatrix} = \begin{bmatrix} -3 & 12 \\ -1 & 4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -4 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \mathbf{x} = t \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

$$x_1 = 4x_2 \qquad t \neq 0$$

$$-2\mathbf{I} - \mathbf{A} = \begin{bmatrix} -2 - 2 & 12 \\ -1 & -2 + 5 \end{bmatrix} = \begin{bmatrix} -4 & 12 \\ -1 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -3 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \mathbf{x} = t \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$$x_1 = 3x_2$$

Eigenvalue problem

Finding eigenvalue and eigenvectors

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 5 & -10 \\ 1 & 0 & 2 & 0 \\ 1 & 0 & 0 & 3 \end{bmatrix}$$

$$|\lambda\mathbf{I} - \mathbf{A}| = \begin{vmatrix} \lambda - 1 & 0 & 0 & 0 \\ 0 & \lambda - 1 & -5 & 10 \\ -1 & 0 & \lambda - 2 & 0 \\ -1 & 0 & 0 & \lambda - 3 \end{vmatrix}$$

$$= (\lambda - 1)^2(\lambda - 2)(\lambda - 3)$$

$\lambda_1 = 1, \ \lambda_2 = 2, \ \lambda_3 = 3$

$$(\mathbf{1})\mathbf{I} - \mathbf{A} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -5 & 10 \\ -1 & 0 & -1 & 0 \\ -1 & 0 & 0 & -2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{x_1} \\ \boldsymbol{x_2} \\ \boldsymbol{x_3} \\ \boldsymbol{x_4} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$x_1 = -2x_4$

$x_3 = 2x_4$

$s = x_2$

$t = x_4$

$$\begin{bmatrix} \boldsymbol{x_1} \\ \boldsymbol{x_2} \\ \boldsymbol{x_3} \\ \boldsymbol{x_4} \end{bmatrix} = \begin{bmatrix} 0s - 2t \\ s + 0t \\ 0s + 2t \\ 0s + t \end{bmatrix} = \boldsymbol{s} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \boldsymbol{t} \begin{bmatrix} -2 \\ 0 \\ 2 \\ 1 \end{bmatrix}$$

$t, s \neq 0$

Eigenvalue problem

Finding eigenvalue and eigenvectors

If A is an n×n triangular matrix, then its eigenvalues are the entries on its main diagonal.

$$\mathbf{A} = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 1 & 0 \\ 5 & 3 & -3 \end{bmatrix}$$

Why do the eigenvalues of a triangular matrix lie along its diagonal?

$$|\lambda\mathbf{I} - \mathbf{A}| = \begin{vmatrix} \lambda - 2 & 0 & 0 \\ 1 & \lambda - 1 & 0 \\ -5 & -3 & \lambda + 3 \end{vmatrix} = (\lambda - 2)(\lambda - 1)(\lambda + 3)$$

$$\lambda_1 = 2, \ \lambda_2 = 1, \ \lambda_3 = -3$$

Eigenvalue problem

Diagonalization

Suppose the n×n matrix **A** has *n* linearly independent eigenvectors $\mathbf{x}_1, \ldots, \mathbf{x}_n$. Put them into the columns of an eigenvector matrix **X**. Then $\mathbf{X}^{-1}\mathbf{A}\mathbf{X}$ is the eigenvalue matrix **Λ**:

$$\mathbf{AX} = \mathbf{A}[\mathbf{x}_1 \ldots \mathbf{x}_n] = [\lambda_1\mathbf{x}_1 \ldots \lambda_n\mathbf{x}_n]$$

$$= \mathbf{X}\begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \lambda_n \end{bmatrix}$$

$$\mathbf{AX} = \mathbf{X\Lambda}$$

$$\mathbf{X}^{-1}\mathbf{AX} = \mathbf{\Lambda}$$

Any matrix that has no repeated eigenvalues can be diagonalized.

Eigenvalue problem

Diagonalization

Show that the matrix A is diagonalizable:

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 5 & -10 \\ 1 & 0 & 2 & 0 \\ 1 & 0 & 0 & 3 \end{bmatrix}$$

$$\lambda_1 = 1, \ \lambda_2 = 2, \ \lambda_3 = 3$$

$$\mathbf{x}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 2 \\ 1 \end{bmatrix} \quad \mathbf{x}_2 = \begin{bmatrix} 0 \\ 5 \\ 1 \\ 0 \end{bmatrix} \quad \mathbf{x}_3 = \begin{bmatrix} 0 \\ -5 \\ 0 \\ 1 \end{bmatrix}$$

$$\mathbf{X} = \begin{bmatrix} 0 & -2 & 0 & 0 \\ 1 & 0 & 5 & -5 \\ 0 & 2 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

X is invertible, so its column vectors form a linearly independent set. This means that A is diagonalizable:

$$\mathbf{X}^{-1}\mathbf{A}\mathbf{X} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

Eigenvalue problem

Steps for Diagonalizing a Square Matrix

Let **A** be an n×n matrix.

1. Find n linearly independent eigenvectors for A (if possible) with corresponding eigenvalues. If n linearly independent eigenvectors do not exist, then A is not diagonalizable.

2. Let **X** be the n×n matrix whose columns consist of these eigenvectors. That is, $X = [\mathbf{x}_1\ \mathbf{x}_2\ .\ .\ .\ \mathbf{x}_n]$.

3. The diagonal matrix $\mathbf{D} = \mathbf{X}^{-1}\mathbf{A}\mathbf{X}$ will have the eigenvalues on its main diagonal.

The order of the eigenvectors used to form **X** will determine the order in which the eigenvalues appear on the main diagonal of **D**.

Eigenvalue problem

Finding eigenvalue and eigenvectors

For large values of *n*, polynomial equations are difficult and time-consuming to solve. Moreover, numerical techniques for approximating roots of polynomial equations of high degree are sensitive to rounding errors.

The power method

- an **iterative** method for approximating eigenvalues
- can be used only to find the **dominant** eigenvalue

The first step is the choose of an **initial approximation** $\mathbf{x}_0$ of one of the dominant eigenvectors of A:

$$\mathbf{x}_1 = \mathbf{A}\mathbf{x}_0$$

Eigenvalue problem

Finding eigenvalue and eigenvectors

$$\mathbf{x}_1 = \mathbf{A}\mathbf{x}_0$$
$$\mathbf{x}_2 = \mathbf{A}\mathbf{x}_1 = \mathbf{A}(\mathbf{A}\mathbf{x}_0)$$
$$\mathbf{x}_3 = \mathbf{A}\mathbf{x}_2 = \mathbf{A}(\mathbf{A}^2\mathbf{x}_0)$$
$$\vdots$$
$$\mathbf{x}_\mathrm{k} = \mathbf{A}\mathbf{x}_{\mathrm{k}-1} = \mathbf{A}\left(\mathbf{A}^{k-1}\mathbf{x}_0\right) = \mathbf{A}^k\mathbf{x}_0$$

For large powers of k, and by properly scaling this sequence, we obtain a good approximation of the dominant eigenvector of **A**.

If **x** is an eigenvector of a matrix A, then its corresponding eigenvalue is given by **Rayleigh quotient**:

$$\lambda = \frac{\mathrm{A}\mathbf{x} \cdot \mathbf{x}}{\mathbf{x} \cdot \mathbf{x}}$$

Provide a Proof for the Convergence of the Power Method

Eigenvalue problem

Finding eigenvalue and eigenvectors

Example: seven iterations of the power method with scaling to approximate a dominant eigenvector of the matrix A:

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 0 \\ -2 & 1 & 2 \\ 1 & 3 & 1 \end{bmatrix} \overset{\mathbf{x}_0}{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}} = \overset{\mathbf{x}_1}{\begin{bmatrix} 3 \\ 1 \\ 5 \end{bmatrix}} \rightarrow \overset{\mathbf{x}_{1\,(scaled)}}{\begin{bmatrix} 3/5 \\ 1/5 \\ 1 \end{bmatrix}}$$

$\mathbf{x}_0$	$\mathbf{x}_1$	$\mathbf{x}_2$	$\mathbf{x}_3$	$\mathbf{x}_4$	$\mathbf{x}_5$	$\mathbf{x}_6$	$\mathbf{x}_7$
$\begin{bmatrix} 1.00 \\ 1.00 \\ 1.00 \end{bmatrix}$	$\begin{bmatrix} 0.60 \\ 0.20 \\ 1.00 \end{bmatrix}$	$\begin{bmatrix} 0.45 \\ 0.45 \\ 1.00 \end{bmatrix}$	$\begin{bmatrix} 0.48 \\ 0.55 \\ 1.00 \end{bmatrix}$	$\begin{bmatrix} 0.51 \\ 0.51 \\ 1.00 \end{bmatrix}$	$\begin{bmatrix} 0.50 \\ 0.49 \\ 1.00 \end{bmatrix}$	$\begin{bmatrix} 0.50 \\ 0.50 \\ 1.00 \end{bmatrix}$	$\begin{bmatrix} 0.50 \\ 0.50 \\ 1.00 \end{bmatrix}$

Scaling factors: 5.00 2.20 2.82 3.13 3.02 2.99 3.00

Eigenvalue problem

Any Question?